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Christian Constanda

# Differential Equations

A Primer for Scientists and Engineers

Second Edition

 Springer

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# Preface to the Second Edition

The second edition of this book contains a number of changes, some of substance and some merely cosmetic. These changes aim to make the book appeal to a wider audience and enhance its user-friendly features.

- A new chapter (Chap. 10) has been added, which presents some basic numerical methods (Euler, Euler midpoint, improved Euler, and fourth-order Runge–Kutta) for approximating the solutions of first-order initial value problems. The exercises in Sects. 10.1–10.3 and the first ten exercises in Sect. 10.4 are based on the same problems, so that students can compare the results produced by the different methods.
- The text of a few sections has been augmented with additional comments, explanations, and examples.
- Appendix B4 has been inserted, to remind users the definition and basic properties of hyperbolic functions.
- The number of worked examples has been increased from 232 to 246.
- The number of exercises has been increased from 810 to 1010.
- All the misprints/omissions detected in the first edition have been corrected.

In Chap. 10, numerical results are rounded to the fourth decimal place. Also, to avoid cumbersome notation, and without danger of ambiguity, the approximate equality symbol has been replaced by the equality sign.

I would like to thank Elizabeth Loew, the executive editor for mathematics at Springer, New York, for her guidance during the completion of this project and, in alphabetical order, Kimberly Adams, Peyton Cook, Matteo Dalla Riva, Matthew Donahue, Dale Doty, William Hamill, Shirley Pomeranz, and Dragan Skropanic for constructive comments and suggestions and for their help with proofreading parts of the manuscript.

Finally, my special gratitude goes to my wife, who accepts that for some mathematicians, writing books is an incurable affliction, whose sufferers are in need of constant support and understanding.

Tulsa, OK, USA  
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Christian Constanda

# Preface to the First Edition

Arguably, one of the principles underpinning classroom success is that the instructor always knows best. Whether this mildly dictatorial premise is correct or not, it seems logical that performance can only improve if the said instructor also pays attention to customer feedback. Students' opinion sometimes contains valuable points and, properly canvassed and interpreted, may exercise a positive influence on the quality of a course and the manner of its teaching.

When I polled my students about what they wanted from a textbook, their answers clustered around five main issues.

*The book should be easy to follow without being excessively verbose.* A crisp, concise, and to-the-point style is much preferred to long-winded explanations that tend to obscure the topic and make the reader lose the thread of the argument.

*The book should not talk down to the readers.* Students feel slighted when they are treated as if they have no basic knowledge of mathematics, and many regard the multi-colored, heavily illustrated texts as better suited for inexperienced high schoolers than for second-year university undergraduates.

*The book should keep the theory to a minimum.* Lengthy and convoluted proofs should be dropped in favor of a wide variety of illustrative examples and practice exercises.

*The book should not embed computational devices in the instruction process.* Although born in the age of the computer, a majority of students candidly admit that they do not learn much from electronic number crunching.

*The book should be "slim."* The size and weight of a 500-page volume tend to discourage potential readers and bode ill for its selling price.

In my view, a book that tries to be "all things to all men" often ends up disappointing its intended audience, who might derive greater profit from a less ambitious but more focused text composed with a twist of pragmatism. The textbooks on differential equations currently on the market, while professionally written and very comprehensive, fail, I believe, on at least one of the above criteria; by contrast, this book attempts to comply with the entire set. To what extent it has succeeded is for the end user to decide. All I can say at this stage is that students in my institution and elsewhere, having adopted an earlier draft as prescribed text, declared themselves fully satisfied by it and agreed that every one of the goals on the above wish list had been met. The final version incorporates several additions and changes that answer some of their comments and a number of suggestions received from other colleagues involved in the teaching of the subject.

In earlier times, mathematical analysis was tackled from the outset with what is called the  $\varepsilon$ - $\delta$  methodology. Those times are now long gone. Today, with a few exceptions, all

science and engineering students, including mathematics majors, start by going through calculus I, II, and III, where they learn the mechanics of differentiation and integration but are not shown the proofs of some of the statements in which the formal techniques are rooted, because they have not been exposed yet to the  $\epsilon$ - $\delta$  language. Those who want to see these proofs enroll in advanced calculus. Consequently, the natural continuation of the primary calculus sequence for all students is a differential equations course that teaches them solution techniques without the proofs of a number of fundamental theorems. The missing proofs are discussed later in an advanced differential equations sequel (compulsory for mathematics majors and optional for the interested engineering students), where they are developed with the help of advanced calculus concepts. This book is intended for use with the first—elementary—differential equations course, taken by mathematics, physics, and engineering students alike.

Omitted proofs aside, every building block of every method described in this textbook is assembled with total rigor and accuracy.

The book is written in a style that uses words (sparingly) as a bonding agent between consecutive mathematical passages, keeping the author's presence in the background and allowing the mathematics to be the dominant voice. This should help the readers navigate the material quite comfortably on their own. After the first examples in each section or subsection are solved with full details, the solutions to the rest of them are presented more succinctly: every intermediate stage is explained, but incidental computation (integration by parts or by substitution, finding the roots of polynomial equations, etc.) is entrusted to the students, who have learned the basics of calculus and algebra and should thus be able to perform it routinely.

The contents, somewhat in excess of what can be covered during one semester, include all the classical topics expected to be found in a first course on ordinary differential equations. Numerical methods are off the ingredient list since, in my view, they fall under the jurisdiction of numerical analysis. Besides, students are already acquainted with such approximating procedures from calculus II, where they are introduced to Euler's method. Graphs are used only occasionally, to offer help with less intuitive concepts (for instance, the stability of an equilibrium solution) and not to present a visual image of the solution of every example. If the students are interested in the latter, they can generate it themselves in the computer lab, where qualified guidance is normally provided.

The book formally splits the "pure" and "applied" sides of the subject by placing the investigation of selected mathematical models in separate chapters. Boundary value problems are touched upon briefly (for the benefit of the undergraduates who intend to go on to study partial differential equations), but without reference to Sturm–Liouville analysis.

Although only about 260 pages long, the book contains 232 worked examples and 810 exercises. There is no duplication among the examples: no two of them are of exactly the same kind, as they are intended to make the user understand how the methods are applied in a variety of circumstances. The exercises aim to cement this knowledge and are all suitable as homework; indeed, each and every one of them is part of my students' assignments.

Computer algebra software—specifically, *Mathematica*<sup>®</sup>—is employed in the book for only one purpose: to show how to verify quickly that the solutions obtained are correct. Since, in spite of its name, this package has not been created by mathematicians, it does not always do what a mathematician wants. In many other respects, it is a perfectly good instrument, which, it is hoped, will keep on improving so that when, say, version 54 is released, all existing deficiencies will have been eliminated. I take the view that to learn mathematics properly, one must use pencil and paper and solve

problems by brain and hand alone. To encourage and facilitate this process, almost all the examples and exercises in the book have been constructed with integers and a few easily managed fractions as coefficients and constant terms.

Truth be told, it often seems that the aim of the average student in any course these days is to do just enough to pass it and earn the credits. This book provides such students with everything they need to reach their goal. The gifted ones, who are interested not only in the *how* but also in the *why* of mathematical methods and try hard to improve from a routinely achieved 95% on their tests to a full 100%, can use the book as a springboard for progress to more specialized sources (see the list under Further Reading) or for joining an advanced course where the theoretical aspects left out of the basic one are thoroughly investigated and explained.

And now, two side issues related to mathematics, though not necessarily to differential equations.

Scientists, and especially mathematicians, need in their work more symbols than the Latin alphabet has to offer. This forces them to borrow from other scripts, among which Greek is the runaway favorite. However, academics—even English speaking ones—cannot agree on a common pronunciation of the Greek letters. My choice is to go to the source, so to speak, and simply follow the way of the Greeks themselves. If anyone else is tempted to try my solution, they can find details in Appendix D.

Many instructors would probably agree that one of the reasons why some students do not get the high grades they aspire to is a cocktail of annoying bad habits and incorrect algebra and calculus manipulation “techniques” acquired (along with the misuse of the word “like”) in elementary school. My book *Dude, Can You Count?* (Copernicus, Springer, 2009) systematically collects the most common of these bloopers and shows how any number of absurdities can be “proved” if such errors are accepted as legitimate mathematical handling. *Dude* is a recommended reading for my classroom attendees, who, I am pleased to report, now commit far fewer errors in their written presentations than they used to. Alas, the cure for the “like” affliction continues to elude me.

This book would not have seen the light of day without the special assistance that I received from Elizabeth Loew, my mathematics editor at Springer–New York. She monitored the evolution of the manuscript at every stage, offered advice and encouragement, and was particularly understanding over deadlines. I wish to express my gratitude to her for all the help she gave me during the completion of this project.

I am also indebted to my colleagues Peyton Cook and Kimberly Adams, who trawled the text for errors and misprints and made very useful remarks, to Geoffrey Price for useful discussions, and to Dale Doty, our departmental *Mathematica*<sup>®</sup> guru. (Readers interested in finding out more about this software are directed to the website [http://www.wolfram.com/mathematica/.](http://www.wolfram.com/mathematica/))

Finally, I want to acknowledge my students for their interest in working through all the examples and exercises and for flagging up anything that caught their attention as being inaccurate or incomplete.

My wife, of course, receives the highest accolade for her inspiring professionalism, patience, and steadfast support.

Tulsa, OK, USA  
April 2013

Christian Constanda

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# Acronyms

<b>DE</b>	Differential equation
<b>IC</b>	Initial condition
<b>BC</b>	Boundary condition
<b>IVP</b>	Initial value problem
<b>BVP</b>	Boundary value problem
<b>GS</b>	General solution
<b>PS</b>	Particular solution
<b>FSS</b>	Fundamental set of solutions