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Music Through Fourier Space

Discrete Fourier Transform in Music Theory

Springer

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ISSN 1868-0305 ISSN 1868-0313 (electronic)
Computational Music Science
ISBN 978-3-319-45580-8 ISBN 978-3-319-45581-5 (eBook)
DOI 10.1007/978-3-319-45581-5

Library of Congress Control Number: 2016954630

Printed on acid-free paper

This Springer imprint is published by Springer Nature
The registered company is Springer International Publishing AG
The registered company address is: Gewerbestrasse 11, 6330 Cham, Switzerland

Introduction

This book is *not* about harmonics, analysis or synthesis of sound. It deals with harmonic analysis but in the abstract realm of musical structures: scales, chords, rhythms, etc. It was but recently discovered that this kind of analysis can be performed on such abstract objects, and furthermore the results carry impressively meaningful significance in terms of already well-known musical concepts. Indeed in the last decade, the Discrete Fourier Transform (DFT for short) of musical structures has come to the fore in several domains and appears to be one of the most promising tools available to researchers in music theory. The DFT of a set (say a pitch-class set) is a list of complex numbers, called Fourier coefficients. They can be seen alternatively as pairs of real numbers, or vectors in a plane; each coefficient provides decisive information about some musical dimensions of the pitch-class set in question.

For instance, the DFT of C♯EGB♭ is

$$(4,0,0,0,4e^{4i\pi/3},0,0,0,4e^{2i\pi/3},0,0,0)$$

where all the 0's show the periodic character of the chord, the sizes of the non-nil coefficients mean that the chord divides the octave equally in four parts, and the angles $(2i\pi/3, 4i\pi/3)$ specify which of the three diminished sevenths we are looking at.

From David Lewin's very first paper (1959) and its revival by Ian Quinn (2005), it came to be known that the *magnitude of Fourier coefficients*, i.e. the length of these vectors, tells us much about the *shape* of a musical structure, be it a scale, chord, or (periodic) rhythm. More precisely, two objects whose Fourier coefficients have equal magnitude are *homometric*, i.e. they share the same interval distribution; this generalization of isometry was initially studied in crystallography. *Saliency*, i.e. a large size of some Fourier coefficients, characterises very special scales, such as the diatonic, pentatonic, whole-tone scales. On the other hand, flat distributions of these magnitudes can be shown to correspond with uniform intervallic distributions, showing that these magnitudes yield a very concrete and perceptible musical meaning. Furthermore, nil Fourier coefficients are highly organised and play a vital role in the theory of tilings of the line, better known as "rhythmic canons."

Finally, the cutting-edge research is currently focused on the other component of Fourier coefficients, their directions (called phases). These phases appear to model some aspects of tonal music with unforeseen accuracy. Most of these aspects can be extended from the discrete to the continuous domain, allowing the consideration of microtonal music or arbitrary pitch, and interesting links with voice-leading theory. This type of analysis can also be defined for ordered collections of non-discrete pitch classes, enabling, for instance, comparisons of tunings.

Historical Survey and Contents

Historically, the Discrete Fourier Transform appeared in D. Lewin's very first paper in 1959 [62]. Its mention at the very end of the paper was as discreet as possible, anticipating an outraged reaction at the introduction of "high-level" mathematics in a music journal – a reaction which duly occurred. The paper was devoted to the interesting new notion of the Intervallic Relationship between two pc-sets[1], and its main result was that retrieval of A knowing a fixed set B and $\mathrm{IFunc}(A,B)$ was possible, provided B did not fall into a hodgepodge of so-called special cases – actually just those cases when at least one of the Fourier coefficients of B is 0. These were the times when Milton Babbitt proved his famous hexachordal theorem, probably with young Lewin's help. As we will see, its expression in terms of Fourier coefficients allows one to surmise that the perception of missing notes (or accents, in a rhythm) completely defines the motif's intervallic structure. These questions, together with any relevant definitions and properties (with some modern solutions to Lewin's and others' problems), are studied in Chapter 1.

Lewin himself returned to this notion in some of his last papers [63], which influenced the brilliant PhD research of I. Quinn, who encountered DFT and especially large Fourier coefficients as characteristic features of the prominent points of his "landscape of chords" [72], see Fig. 4.1. Since he had voluntarily left aside for readers of the *Journal of Music Theory* the 'stultifying' mathematical work involved in the proof of one of his nicer results, connecting Maximally Even Sets and large Fourier coefficients, I did it in [10], along with a complete discussion of all maxima of Fourier coefficients of all pc-sets, which is summarised and extended in Chapter 4. Lacklustre Fourier coefficients, with none showing particular saliency, are also studied in that chapter.

Meanwhile, two apparently extraneous topics involved a number of researchers in using the very same notion of DFT: homometry which is covered in Chapter 2 (see the state of the art in [2, 64] and Tom Johnson's recent compositions *Intervals* or *Trichords et tetrachords*); and rhythmic canons in Chapter 3 – which are really algebraic decompositions of cyclic groups as direct sums of subsets. The latter can be used either in the domain of periodic rhythms or pitches modulo some 'octave,' and were first extensively studied by Dan Tudor Vuza [94][2], then connected to the general

[1] I use the modern concept, though the term 'pitch-class set' had not yet been coined at the time. $\mathrm{IFunc}(A,B)$ is the histogram of the different possible intervals from A to B.

[2] At the time, probably the only theorist to mention Lewin's use of DFT.

theory of tiling by [19, 17] and developed in numerous publications [8, 18, 73] which managed to interest some leading pure mathematician theorists in the field (Matolcsi, Kolountzakis, Szabó) in musical notions such as Vuza canons.[3]

There were also improbable cross-overs, like looking for algebraic decompositions of pc-collections (is a minor scale a sum and difference of major scales?) [13], or an incursion into paleo-musicology, quantifying a quality of temperaments in the search for the tuning favoured by J.S. Bach [16], which unexpectedly warranted the use of DFT.

Aware of the intrinsic value of DFT, several researchers commented on it, trying to extend it to continuous pitch-classes [25] and/or to connect its values to voice-leadings [89, 88]. These and other generalisations to continuous spaces are studied in Chapter 5. Another very original development is the study of all Fourier coefficients with a given index of all pc-sets [50], also oriented towards questions of voice-leadings. On the other hand, consideration of the profile of the DFT enables characterisation of pc-sets in diverse voices or regions of tonal and atonal pieces [98, 99] as we will see in Chapter 6, which takes up the dimension that Quinn had left aside, the *phase* (or direction) of Fourier coefficients. The position of pairs of phases (angles) on a torus was only recently introduced in [15] but has known tremendously interesting developments since, for early romantic music analysis [96, 97] but also atonal compositions [98, 99]. Published analyses involve Debussy, Schubert, Beethoven, Bartok, Satie, Stravinsky, Webern, and many others. Other developments include, for instance, comparison of intervals inside chromatic clusters in Łutoslawski and Carter, using DFT of pitches (not pitch classes) by Cliff Callender [25].

A Couple of Examples

I must insist on the fact that DFT analysis is no longer some abstract consideration, but is done on actual music: consider for instance Chopin's *Etude op. 10, N°*5, wherein the pentatonic (black keys) played by the right hand is a subset of G♭ major played by the left hand; but so are many other subsets (or oversets). I previously pointed out in [10] that, because the pentatonic and diatonic scales are complementary Maximally Even Sets, one is included in the other up to transposition (warranting the name 'Chopin's Theorem' for this property of ME sets); however, it is much more significant to observe that these two scales have *identical Fourier coefficients with odd indexes*[4], which reflects spectacularly their kinship (see Chapter 6 and Fig. 4.7). I cannot wait to exhibit another spectacular example of the 'unreasonable efficiency' of DFT: Jason Yust's discovery [98] that in Bartok's *String Quartet 4* (iv), the accompaniment concentrates its energy in the second Fourier component while this component vanishes for the melody, and conversely for the sixth component (associated with the whole-tone character). This is again vastly superior to classic

[3] The musical aspect lies in the idea that a listener does not hear any repetition either in the motif nor in the pattern of entries of a Vuza canon.

[4] The other coefficients, with even indexes, have the same magnitude, but different directions.

'Set-Theory' subset-relationships (parts of this analysis and others are reproduced in this book), cf. Fig. 0.1 (further commented on in Chapters 4 and 6).

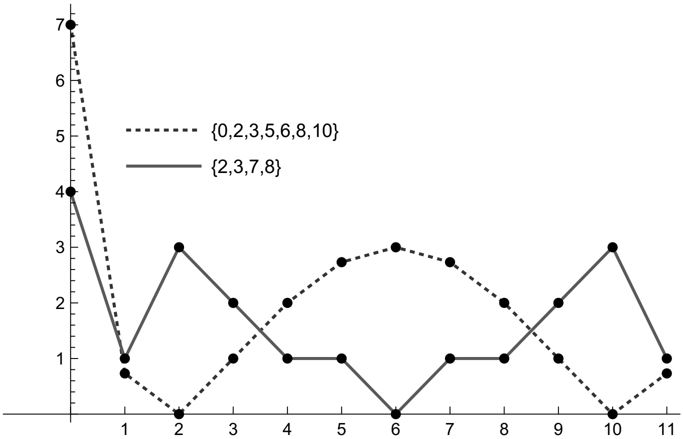

Fig. 0.1. DFT magnitudes of melody and accompaniment in Bartok

One explanation of the efficiency of DFT in music theory may well be Theorem 1.11. As we will see throughout this book, many music theory operations can be expressed in terms of *convolution products*. Not only is this product significantly simpler in Fourier space (i.e. after Fourier transform, cf. Theorem 1.10), but the aforementioned theorem proves that **Fourier space is the only one where such a simplification occurs**. This means that, for instance, interval functions or vectors, which are essential in the perception of the shape of musical objects, are more easily constructed and even perceived in Fourier space. *Idem* for the property of tiling – filling the space with one motif according to another – which is completely obvious when glancing at nil Fourier coefficients. Furthermore, we will see and understand how each and every polar coordinate in Fourier space carries rich musical meaning, not requiring any further computation.

Public

This book aims at being self-contained, providing coherent definitions and properties of DFT for the use of musicians (theorists and practitioners alike). A wealth of examples will also be given, and I have chosen the simplest ones since my purpose is clarity of exposition. More sophisticated examples can be found in the already abundant bibliography. I have also added a number of exercises, some with solutions, because the best way to make one's way through new notions is always with pen and pencil.

Professional musicians, researchers and teachers of music theory are of course the privileged public for this monograph. But I tried to make it accessible at pregraduate level, either in music or in mathematics. In the former case, besides introducing the notion of DFT itself for its intrinsic interest, it may help the student progress through useful mathematical concepts that crop up along the way. In the latter case, I hope that maths teachers may find interesting material for their classes, and that the musical angle can help enlighten those students who need a purpose before a concept. It is even hoped, and indeed expected, that hardened pure mathematicians will find in here a few original results worth their mettle.

Some general, elementary grounding in mathematics should be useful: knowledge of simple number sets (integers, rationals, real and complex numbers), basics of group theory (group structure, morphism, subgroups) which are mostly applied to the group $\mathbb{Z}_{12}$ of integers modulo 12; other simple quotient structures make furtive appearances in Chapters 1 and 3; vector spaces and diagonalization of matrixes are mentioned in Chapter 1 and used once in Chapter 2, providing sense to the otherwise mysterious 'rational spectral units'. The corresponding Theorem 2.10 is the only really difficult one in this book: many proofs are one-liners, most do not exceed paragraph length. All in all, I hope that any cultured reader with a smattering of scientific education will feel at ease with most of this book (and will be welcome to skip the remaining difficulties). On the other hand, mathematically minded but non-musician readers who cannot read musical scores or are unfamiliar with 'pc-sets' or 'scales' can rely on the omnipresent translations into mathematical terms.

Last but not least, some online content has been developed specifically for the readers of this book, who are strongly encouraged to use it: for instance all 'Fourier profiles' of all classes of pc-sets can be perused at
`http://canonsrythmiques.free.fr/MaRecherche/photos-2/`
while only a selection of the 210 cases is printed in Chapter 8, and software is available for the computation of the DFT of any pc-set in $\mathbb{Z}_{12}$.

Acknowledgements

First and foremost, my gratitude to Ian Quinn, who revived interest in DFT and invented the saliency quality. He is, even more than Lewin, simultaneously father and midwife of this new sub-discipline. May he be praised forever for this invaluable step forward.

I am very much indebted to Jason Yust, who made tremendous progress in the field in the last two years and generously gave me permission to cite all of his results and analyses, even those not yet published.

Jack Douthett is also father to a prolific notion, the Maximally Even Sets, which are a foundation to many further developments, including the present book. His support and encouragements were always a great help in my research.

I am grateful to Moreno Andreatta and Guerino Mazzola, who incited me to write this book and provided pointed and vital advice.

Among several memorable research collaborators or co-authors – Carlos Agon, Moreno Andreatta, Daniel Ghisi, John Mandereau, Thomas Noll – I would like to single out for the present opus William Sethares, since our joint work on matricial shortcuts through music-theoretical notions provided some major insights on the usefulness of DFT.

I have used many times a canon by composer Georges Bloch and I am grateful for his permission. Cliff Callender allowed me to borrow from the well-chosen examples of his paper on Fourier; his openness and helpfulness are as usual greatly appreciated.

Tom Johnson has been a constant source of stimulation in my research. He also provided many compositions of interest for this book, which would have been much more terse without him. He proves every day that these mathematical speculations of ours pave the way to making very real music.

I remember with pleasure the fruitful discussions on frequency of interval classes vs. Fourier saliency that were exchanged with Aline Honingh. It influenced the overall shaping of Chapter 4.

The music and maths community, organised around the SMCM conference and the *Journal of Mathematics and Music*, has been since its foundation a constant and stimulating source of inspiration. I would like to cite all its members.

IMSLP is an invaluable source for free-of-rights musical scores, from which I borrowed much more than appears in the printed book.

Many thanks to my proof-readers: my children Jeanne and Raphaël, Hélianthe Caure and especially lynx-eyed Serge Bastidas, who spotted obscure misprints even in the maths. Jeanne and my niece Cora were a great help in enhancing my homemade graphics. The professionalism of the Springer team and its readings is unrivaled.

Last but certainly not least, to my wife, Pascale, whose patience was several times severely tested during the elaboration of this book.

Saint Nazaire, P.O., France, *Emmanuel Amiot*
October 2014 - May 2016.

Notations

- Sets are given between curly brackets: $\{0,4,7\}$. Sequences or n-uplets, taking into account the order of elements, use parentheses: $(0,7,4)$.
- $[a,b], [a,b[$ are respectively closed and semi-open intervals. For intervals of integers I use $[\![2,5]\!] = \{2,3,4,5\}$.
- $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$ are respectively the sets of natural integers, integers, rationals, real and complex numbers.
- $a \mid b$ means that a is a divisor of b (most of the time a, b will be integers, in a few occasions I will use divisors of polynomials).
- $\mathrm{Div}(n)$ is the set of divisors of n: $\mathrm{Div}(12) = \{1,2,3,4,6,12\}$. The greatest common divisor is denoted by $\gcd(a,b)$.
- $\varphi(n)$ is Euler's totient function. Several definitions will be provided in this book.
- $|z|$ is the absolute value, or magnitude, of a real or complex number z.
- $\#A$ is the cardinality of the set A, i.e. its number of elements.
- $\mathbf{1}$ denotes the constant map with value 1. Any constant maps can thus be described as $c.\mathbf{1}$.
- $\mathbb{Z}_n$ is short for $\mathbb{Z}/n\mathbb{Z}$, the cyclic group (or ring) with n elements. 'Pitch-classes' (i.e. notes modulo octave equivalence) are modeled by the elements of $\mathbb{Z}_{12}$, or $\mathbb{Z}_n$ if the octave is divided into n parts. 'Pitch-class sets' or 'pc-sets' are subsets of $\mathbb{Z}_n$.
- More generally, the slash / denotes a quotient structure: $\mathbb{R}/(2\pi\mathbb{Z})$ means real numbers modulo any multiple of 2π, i.e. angles. In a few situations, more complex quotient structures are used (say $\mathbb{Z}[X]/(X^n - 1)$, i.e. a ring of polynomials modulo an ideal) and will be explained on the fly.
- T/I is the dihedral group (usually in $\mathbb{Z}_{12}$) whose elements are the transpositions (translations) $x \mapsto x + c$ and inversions (central symmetries) $x \mapsto -x + c$.
- Equality modulo some n is written $a \equiv b \pmod{n}$. In a few places, I will state polynomial equations modulo $X^n - 1$, meaning that all powers of X have their exponents reduced modulo n (e.g. $X^{3n+2} \equiv X^2$).
- Abbreviation 'iff' stands for 'if and only if', sometimes the symbol $\Longleftrightarrow$ will be used.
- The symbol $\approx$ is used for isomorphisms (ex: $\mathbb{Z}_3 \times \mathbb{Z}_4 \approx \mathbb{Z}_{12}$). It is also used for approximate values of numbers, without risk of confusion ($\pi \approx 3.14$).
- $A \setminus B$ is the set of elements of A which are not elements of B.
- $\mathbb{Z}_n^*$ is the multiplicative group of invertible elements in $\mathbb{Z}_n$, i.e. the classes of integers coprime with n. Unless n is prime, this is not to be confused with $\mathbb{Z}_n$ deprived of 0, i.e. $\mathbb{Z}_n \setminus \{0\}$: for instance, $\mathbb{Z}_{12}^* = \{1,5,7,11\}$. Similarly $K \setminus \{0\}$ means that 0 is omitted in set K.
- Direct products of structures ($\mathbb{Z}_3 \times \mathbb{Z}_4$) and direct sums ($\{0,3,6,9\} \oplus \{0,4,8\} = \mathbb{Z}_{12}$) will be used freely. There will be very few occurrences of *semi-direct products*, e.g. $\mathbb{Z}_{12} \rtimes \mathbb{Z}_2$, that the reader is welcome to skip if unfamiliar with this notion.
- In Chapter 6, I sometimes use the notation $t = 10, e = 11$ for readability.

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