Group Representation for Quantum Theory

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Preface

This book is the English edition of the Japanese book *Group Representations for Quantum Theory*, which was originally published by Kyoritsu shuppan, Tokyo, Japan in January 2014. The original Japanese book covers several topics in representation theory that is related to quantum theory. As is well known, group representation theory is a very strong tool for quantum theory, in particular, angular momentum, hydrogen-type Hamiltonian, spin–orbit interaction, quark model, quantum optics, and quantum information processing including quantum error correction. Therefore, many departments of physics have lecture courses for mathematics for physics, in particular, they have a graduate lecture course for representation theory for physics. This book conducts lecture courses on mathematics for physicists. When the contents of this book are too much for the lecture course, this book can be used by skipping several detailed parts.

To describe a big picture of application of representation theory to quantum theory, the book needs to contain the following six topics, permutation group, SU(2) and SU(d), Heisenberg representation, squeezing operation, Discrete Heisenberg representation, and the relation with Fourier transform from a unified viewpoint by including projective representation. Unfortunately, although there are so many good mathematical books for a part of six topics, no book contains all of these topics because they are too segmentalized. Further, some of them are written in an abstract way in mathematical style and, often, the materials are too segmented. At least, the notation is not familiar to people working on quantum theory. Others are good elementary books, but do not deal with topics related to quantum theory. In particular, such elementary books do not cover projective representation, which is more important in quantum theory. On the other hand, there are several books for physicists. However, these books are too simple and lack the detailed discussion. Hence, they are not useful for advanced study even in physics.

To resolve this issue, the author published the Japanese version of this book. It starts with the basic mathematics for quantum theory. Then, it proceeds to the foundation of group representation theory by including finite group. During this discussion, this book deals with mathematical formulation of boson and fermion, which are fundamental concepts in quantum theory. After this preparation, it discusses representation theory of Lie group and Lie algebra. Based on these mathematics, this book addresses bosonic system and its discretization, which are more related to quantum optics and quantum information, respectively.

However, the original Japanese version has less application to quantum physics. The author has newly added Chap. 5, the later half of Chap. 1, crucial contents of Sect. 4.4, and the discussion for Wigner functions in Chap. 7 as follows. Originally, Chap. 1 discusses only the minimum preparation of mathematics of quantum theory. For smooth connection from basic mathematics to representation theory, the author has added a section for Hamiltonian, which explains the role of Hamiltonian and the relation to representation theory. Also, the author has added a section for symmetry, which describes the details of roles of representation theory in quantum theory. Additionally, the author has added a section for the unbounded case, which is applied only to the infinite dimensional case and can be skipped when the reader is not interested in the difficulty of the infinite-dimensional case. Originally, Sect. 4.4 briefly explained boson and fermion based on the representation theory for Lie group and Lie algebra. In this English version, the author has added several helpful examples including spin, which clarify the need for representation theory for understanding boson and fermion. Unfortunately, many existing books explain boson and fermion without group representation. The author believes that this section is helpful for the study of boson and fermion on finite-dimensional system.

Chapter 5 is devoted to applications of the representation theory to physical systems. This additional chapter starts with the spectral decomposition of the Hamiltonian on 3-dimensional physical system with a rotational invariant potential by using the group $\mathrm{SO}(3,\mathbb{R})$. As a generalization of a part of this topic, we proceed to the spectral decomposition of the Laplacian on the general dimensional sphere, which is essentially the same as the Hamiltonian. Hydrogen-type atom is a special case of 3-dimensional physical system with a rotational invariant potential. The Hamiltonian of this case has a hidden symmetry of $\mathrm{SO}(4,\mathbb{R})$. Indeed, this Hamiltonian has a large degeneracy, which cannot be explained by the visible symmetry of $\mathrm{SO}(3,\mathbb{R})$. The hidden symmetry of $\mathrm{SO}(4,\mathbb{R})$ explains such a large degeneracy. Then, it proceeds to the spin-orbit interaction, in which, the irreducible decomposition of group representation on the composite system (Clebsch Gordan coefficient) plays an important role. This kind of system frequently appears in spin magnetics. Finally, this chapter addresses quark model, which is the key concept of fundamental particle theory. The discussion on the quark model is composed of the analysis on finite-dimensional system, which can be regarded as a good exercise of fermion and group representation theory of a finite-dimensional system. To discuss quark model precisely, we need to discuss the notion of flavor, spin, and color, simultaneously. Although we need many preparations to discuss the whole topic thoroughly and consistently, this book covers all the essential preparation. We also note that several books on the quark model discuss the subject inconsistently due to a lack of basic preparation at the beginning of the books. In particular, this book explicitly writes down the wave functions of all of Baryon and Meson with $\mathrm{SU}(3)$ symmetry while they are not given in many basic books for particle physics. This description will be helpful for students of particle physics. In the final section of Chap. 5, we discuss uncertainty relation for wake packets on various spaces, i.e., not only on the set of real numbers $\mathbb{R}$ but also on the one-dimensional and three-dimensional spheres S^1 and S^3.

Additionally, the author has newly added recent progresses for design theory as discretization of Lie group and homogeneous space in Sect. 4.5 and recent progresses for mutually unbiased bases (MUB) and symmetric informationally complete (SIC) vectors in Sect. 8.4. Another unique feature of this book is the clarification on the relation between group representation and Fourier analysis in Sects. 2.8 and 3.8. Fourier analysis is another important mathematical structure of quantum physics. In this revision, based on this relation, in Chap. 7, the author has added the description for Winger function, which is a key concept for the duality between the position and the momentum in foundation of quantum theory. Then, we discuss the uncertainty relation for Winger function. Since the uncertainty relations on the one-dimensional and three-dimensional spheres S^1 and S^3 and on Winger function were obtained recently, these topics have not been discussed in other books. So, the reader can understand the duality in the phase space more deeply.

Further, for better understanding, the author has added many figures, tables, and exercises to help the reader to understand the materials better so that this book contains 54 figures, 23 tables, and 111 exercises with solutions. This book is organized as follows. First of all, when the section or the example is too advanced, the symbol * is indicated in the title. Since such parts will be used only in the later parts with the symbol *, the reader can still understand the contents well even if the reader omit them in the first reading. The author recommends a beginner of representation theory to omit these parts in the first time. Since the symbols of representations are too complicated, this book summarizes the symbols as several tables (Tables 1-8). The reader can refer the tables to recall the symbols.

We now describe the whole structure of this English edition. As the author explained before, this book treats projective representation as well as conventional representation. Chapter 1 starts the mathematical preparation for quantum theory, and explains the back ground of group representation theory. Chapter 2 introduces group representation theory with fundamental concepts. Chapter 3 deals with general theories that do not depend on the types of Lie groups and Lie algebras. Then, it introduces the Fourier transform for Lie groups. Chapter 4 treats representations of special Lie groups and special algebras, $\mathrm{SU}(2)$, $\mathrm{SU}(1,1)$, and $\mathrm{SU}(d)$. Based on them, we discuss boson and fermion as indistinguishable particles in Sect. 4.4. We also discuss the coherent states, which are not discussed in the conventional textbooks. Using these preparations, Chap. 5 proceeds the above-mentioned applications. As advanced topics, Chap. 6 deals with representations of Lie group and Lie algebra with general form based on a root system including representations of non-compact Lie group. Since these topics are more advanced, the proofs of many theorems are omitted. The reader may need to spend a long time to understand some of these proofs. This chapter summarizes such advanced results very reasonably so that a reader can grasp the contents intuitively based on analogies with simple cases, which is another advantage of this book.

Chapter 7 deals with Heisenberg representation, which gives the bosonic system and plays an important role in quantum optics. Using the bosonic system, this chapter explains second quantization, which is a key concept of quantum field theory. In the end of Chap. 7, we discuss multi-mode squeezing, which requires knowledge for representations of Lie group and Lie algebra based on a root system, which are explained in Chaps. 3, 4, and 6. Chapter 8 deals with discrete Heisenberg representation as a discrete version of Heisenberg representations addressed in Chap. 7. This representation is useful in quantum information, especially, quantum error correcting codes, and designs of quantum circuits. Since any quantum information process is constructed based on a combination of quantum circuits, designs of quantum circuits are crucial for quantum information. Chapter 8 is organized so that the contents can be understood with the contents of Chaps. 1 and 2.

Finally, the author emphasizes the difference from existing books for representation theory as follows. There are so many books for representation theory. At least, there exist several good books containing a part of the contents of this book, in particular, the major parts of Chaps. 3, 4, and 6. However, no book contains the whole contents of this book. Further, many existing mathematical books do not adopt the notation familiar to physicist. Mathematical books containing the detail of projective representation are often too advanced. Typically, such books for representation theory do not explain the relation between the representation theory and the foundation of quantum theory, e.g., boson, fermion, second quantization, Winger function, and quantum circuits. Indeed, since representation theory requires too complicated notations, students have trouble to interpolate notations across several books by themselves. To resolve this problem, they need a single book that incorporates representation theory, which brings them a big picture of quantum theory. Therefore, the author believes that this book is helpful for students for representation theory for quantum theory.

The author is grateful when the readers would be interested in representation theory and quantum theory via this book. Finally, the author expresses the acknowledgments to all persons who cooperate to this English version. Especially, the author would like to thank Prof. Hideyuki Ishi of Nagoya University, Prof. Kwek Leong Chuan of Nanyang Technological University and National University of Singapore, Prof. Serge Richard of Nagoya University, Dr. Huangjun Zhu of University of Cologne, Prof. Soichi Okada of Nagoya University, Prof. Hiroyuki Kanno of Nagoya University, Prof. Toru Uzawa of Nagoya University, and Café-David, which is mathematics salon in Graduate School of Mathematics, Nagoya University for providing helpful comments for this English version. The author would also like to thank Dr. Claus E. Ascheron of Springer Science+Business Media for his encouragement and patience during the preparation of the manuscript.

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Preface to the Japanese Version

Group representational symmetry is one of most fundamental concepts in quantum theory, and has been applied to various areas in physics, e.g., particle physics, nuclear physics, condensed matter physics, and statistical physics. It also plays an important role in quantum information, which enables us fruitful information processing by using quantum phenomena. Especially, since various types of representations of various groups appear in various quantum systems, group representation theory reveals so many aspects of quantum theory. Unfortunately, besides various topics underlie group representation theory, such relations are not sufficiently recognized. In fact, such a recognition often leads us not only to deeper understanding of the topic but also to generalization of the topic. However, since the methods of operator algebra and partial differential equation have been emphasized in the area of mathematical foundation of quantum theory, that of group representation has not taken a sufficiently prominent position in this area. Hence, few books cover various methods in representation theory in a unified viewpoint. On the other hand, representation theory has been significantly developed in various directions as a part of mathematics. However, such fruitful developments are not accessible for students and researchers of quantum theory due to the problems explained later.

As useful knowledge of representation theory for quantum theory is divided into various subtopics in representation theory, they are so segmentalized that no mathematical book provides a simple collection of such knowledges from a unified viewpoint. If we deal with representation theory from mathematics, we usually specify an individual topic of representation theory. As its own characteristic features, representation theory inevitably employs very complicated symbols. When we address a topic across distinct books, we need to spend much effort to adjust the different notations among these books. Especially, representations of real Lie groups play an important role in quantum theory, and they are classified via representations of real Lie algebras. However, since many mathematical introductory books are written based on representations of complex Lie algebras, only a few mathematical introductory books emphasize real Lie algebras. In fact, although representations of real Lie algebras are obtained from representations of complex

Lie algebras via a suitable conversion, few introductory books describe this conversion carefully. In particular, many mathematicians avoid to handle representations of real Lie algebras with the above conversion because it requires more complicated notations while it is not mathematically interesting. Since there are so many elegant mathematical introductory books for individual subtopics of representation theory in the above way, we need a book to connect these individual books while few mathematical introductory books carefully describe such connection parts.

On the other hand, many books in physics care such a point; however, they omit their descriptions so much that the reader cannot understand the contents precisely. Further, projective representations play an central role in quantum theory, but many mathematical books do not cover them sufficiently. Especially, major aspects of projective representation can be treated by trivial extension, but projective representation has several blind points that requires special treatment different from conventional representation. Indeed, such subtle points are linked to variety of quantum phenomena. As another problem, many mathematical books are written in a too generalized form, and do not explain examples related to quantum theory. Also, their descriptions are far from the description of quantum theory. There are so many factors that inhibit researchers of quantum theory from accessing useful results of representation theory in this way.

The contents of this book are composed of mathematical knowledge for representation theory that are well-known for many mathematicians, and reorganized by using notations of quantum theory so that they can be easily applied to various topics in quantum system from the viewpoint of quantum theory. Since this book deals with various topics of representation theory essential for quantum system, the whole structure of representation theory will be clarified from quantum theory. This book emphasizes the similarities among several related topics so that their analogies enable the readers to easily understand some complicated topics based on related simpler topics. Hence, the reader will grasp the key points of these advanced topics of representation theory easily. That is, this book will work as a basic infrastructure to understand quantum theory from representation theory. Since the author studies quantum information mainly, the contents are related to applications to quantum information. However, as quantum information is related to foundation of quantum theory, they will be useful for understanding quantum theory beyond quantum information. Therefore, once the readers complete to read this book, they will be able to understand quantum theory much more deeply based on the representation theory. Also, they can proceed to read another book "A Group Theoretic Approach to Quantum Information."

This book is organized as follows. First of all, when the section or the example is too advanced, the symbol * is indicated in the title. Since such parts will be used only in the latter parts with the symbol *, the reader can understand the contents except for such parts even if the reader omits them. The author recommends a beginner of representation theory to omit these parts in the first time. Since the symbols of representations are too complicated, this book summarizes the symbols as several tables (Tables 1–8). The reader can refer the tables to remember the symbols.

Chapter 1 introduces basic concepts of quantum theory, measurement, state, composite system, and entanglement. It also prepares mathematical notations for quantum systems. Although these notations are specified to quantum systems, they are helpful for group representation. Hence, this book consistently deals with representation theory based on these notations. Although the second quantization is an important topic in quantum field theory, we explain it in Chapter 6 (new Chapter 7) because it needs the Bosonic system.

Chapter 2 discusses representations for group in a general framework including projective representations, which are important in quantum theory. Since projective representation is closely related to extension of group, this chapter focuses on this relation. To discuss representation theory including projective representations, we need to handle the factor system, i.e., the set of phase factors, which requires complicated notations. Since the discussions with projective representations are complicated and do not seem essential for mathematics, many mathematically standard textbooks omit them. However, since such discussions are essential for quantum theory, this chapter handles projective representations by using factor systems and we keep this style during the whole book, which is a distinct point from other related books. This chapter proceeds to the details of representations of finite groups so that it introduces the Fourier transform for finite groups, which connects analysis and algebra. As a typical example, we analyze representations of a permutation group by using Young diagrams.

Chapters 3, 4, and 5 address representation theory for Lie groups and Lie algebras. Especially, Chapter 3 deals with general theories that do not depend on the types of Lie groups and Lie algebras. Chapter 3 treats projective representations of Lie groups and Lie algebras by combining the contents of Chapter 2 although few traditional introductory textbooks touch them. Then, it introduces the Fourier transform for Lie groups including the case of projective representations. It also prepares several concepts for Chapter 6 (new Chapter 7). Also, Chapter 3 introduces complex Lie groups and complex Lie algebras.

Chapter 4 treats representations of special Lie groups and special algebras. Since representations of a Lie algebra can be classified with maximum weight, those of a Lie group can be easily treated through those of the corresponding Lie algebra. Also, Lie algebras provide several concepts important for quantum theory. Hence, Chapter 4 is organized so that it constructs representation of a Lie group via that of the corresponding Lie algebra. Chapter 4 starts with representations of Lie algebras $\mathfrak{su}(2)$ and $\mathfrak{su}(1,1)$. Because the Lie algebra $\mathfrak{su}(2)$ is compact and the Lie algebra $\mathfrak{su}(1,1)$ is not compact, they require different treatment caused in this difference. As they have unexpected common features, we handle both in a unified way. Then, we proceed to representation of the Lie algebra $\mathfrak{su}(r)$ by using Young diagrams. Especially, the representation of the Lie algebra $\mathfrak{su}(r)$ on the tensor product space is closely related to that of the permutation group on the same tensor product space. The relation is called Schur duality. We also consider how a finite subgroup can take a role of the Lie group when its representation is given. Such a problem is called design, and is discussed in this chapter.

Chapter 5 (new Chapter 6) deals with representations of a Lie group via those of the corresponding Lie algebra. This chapter discusses a noncompact Lie group as well as a compact Lie group. To discuss both, we focus on the relation between a (real) Lie algebra and a complex Lie algebra. Since representations of a noncompact group are very complicated, this chapter treats only a part of its representations that are related to quantum theory. Since such a special representation has analogies with representations of a compact Lie group, they can be more easily understood than the general case. As this chapter is composed of very advanced topics and such sections are labeled with *, The author recommends the reader to omit this chapter in the first time. Indeed, this book is organized so that the reader can understand the large part even though such advanced parts are omitted.

Chapter 6 (new Chapter 7) deals with Heisenberg representation, which gives the Bosonic system. Since Heisenberg representation is projective representation, the general theory for projective representation given in Chapter 3 plays an essential role. It also treats the representation of $\mathfrak{su}(1,1)$ that describes the squeezing operations. We discuss the multi-mode Bosonic system as well as the one-mode Bosonic system. Such a quantum system is called the continuous system, and attracts attention because it can be easily implemented in a particular sense. Further, using the Bosonic system, we explain the second quantization and the physical meaning of the tensor product state. Only sections with symbol * require knowledge given in the latter part of Chapter 5.

Chapter 7 (new Chapter 8) deals with discrete Heisenberg representation as a discrete version of the representations addressed in Chapter 6 (new Chapter 7). Although discrete Heisenberg representation is usually constructed from finite fields, there is another type of discrete Heisenberg representation based on the commutative algebra $\mathbb{Z}_d$. So, this book covers both types of discrete Heisenberg representations. Further, we address a discrete version of the representation corresponding to the squeezing operation given in Chapter 6 (new Chapter 7). This representation is closely related to quantum circuits and quantum error correction. While Chapter 7 (new Chapter 8) is related to Chapter 6 (new Chapter 7), this chapter can be read independently of Chapter 6 (new Chapter 7). Since it addresses representations of finite groups, it can be read after Chapter 2. The final section discusses mutually unbiased bases (MUB) and symmetric informationally complete (SIC) vectors, which have several applications.

In the above-mentioned way, this book summarizes knowledges of representation theory that is useful for quantum theory. Then, based on these, this book also explains the foundation of quantum theory, e.g., second quantization and quantum circuits. The author is grateful when the readers would be interested in representation theory and quantum theory via this book.

Finally, the author expresses the acknowledgments to all persons who cooperate to this book. Especially, the author would like to thank Prof. Hideyuki Ishi of Nagoya University, Dr. Wataru Kumagai of Kanagawa university, Prof. Akito Hora of Hokkaido University, and Dr. Tsuyoshi Miezaki of Yamagata University for providing helpful comments. The author would also like to thank Mr. Hideo Kotobuki and Ms. Kei Akagi of Kyoritsu Shuppan for their supports.

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About the Author

Masahito Hayashi was born in Japan in 1971. He received his B.S. degree from the Faculty of Sciences in Kyoto University, Japan, in 1994 and his M.S. and Ph.D. degrees in Mathematics from Kyoto University, Japan, in 1996 and 1999, respectively.

He worked in Kyoto University as a Research Fellow of the Japan Society of the Promotion of Science (JSPS) from 1998 to 2000, worked in the Laboratory for Mathematical Neuroscience, Brain Science Institute, RIKEN from 2000 to 2003, worked in ERATO Quantum Computation and Information Project, Japan Science and Technology Agency (JST) as the Research Head from 2000 to 2006. He also worked in the Superrobust Computation Project Information Science and Technology Strategic Core (21st Century COE by MEXT) Graduate School of Information Science and Technology, The University of Tokyo as Adjunct Associate Professor from 2004 to 2007. He worked in the Graduate School of Information Sciences, Tohoku University as Associate Professor from 2007 to 2012. In 2012, he joined the Graduate School of Mathematics, Nagoya University as Professor. He also worked in the Centre for Quantum Technologies, National University of Singapore as Visiting Research Associate Professor from 2009 till now. In 2011, he received the Information Theory Society Paper Award (2011) for Information-Spectrum Approach to Second-Order Coding Rate in Channel Coding. In 2016, he received the Japan Academy Medal from the Japan Academy and the JSPS Prize from Japan Society for the Promotion of Science.

He is a member of the Editorial Board of the International Journal of Quantum Information and International Journal on Advances in Security. His research interests include classical and quantum information theory, information-theoretic security, and classical and quantum statistical inference.

Symbols

Table 1 Symbols for groups (see Tables 3.1, 3.2, and 3.3 for other groups)

Symbol	Name (Definition)	Page
$H \times K$	Direct product group of groups H and K	28
$H \rtimes K$	Semi direct product of groups H and K	28
$H(K : H)$	Group of extension of group H by group K	34
Ker f	Kernel of homomorphism f	27
$[G, G]$	Commutator subgroup of G	27
$\mathrm{H}(2r, \mathbb{Z}_d)$, $\mathrm{H}(2r, \mathbb{F}_q)$	Discrete Heisenberg group of degree r	35, 36
$C_G(M)$	Centralizer of subgroup M of group G	26
$C(G)$	Center of G ($C_G(G)$)	26
$N_G(M)$	Normalizer of M	26
$\mathrm{U}(1)$	Unit circle	23
U_k	Subgroup $\{e^{i2j/k} \| j \in \mathbb{Z}\}$ of $\mathrm{U}(1)$	23
S_n	Permutation group of degree n	23
$\mathrm{GL}(\mathcal{H})$	General linear group on $\mathcal{H}$	37
$\mathrm{U}(\mathcal{H})$	Unitary group on $\mathcal{H}$	37
$\mathrm{SU}(\mathcal{H})$	Special unitary group on $\mathcal{H}(\mathrm{U}(\mathcal{H}) \cap SL(\mathcal{H}))$	37
G_0	Connected component of G including identity element e	70
$\pi_1(G)$	Fundamental group	95
$\tilde{G}$	Extension of commutative group C by G (universal covering group when C is $\pi_1(G)$)	46, 47, 98
$\breve{G}$	Double covering group of G	97
$\overline{G}_\omega$	Extension of group G by $\mathrm{U}(1)$ defined by $\in C(G)$	49
$\mathrm{Sp}(2r, \mathbb{F}_q)$, $\mathrm{Sp}(2r, \mathbb{Z}_d)$	Discrete symplectic group	273

(continued)

Table 1 (continued)

Symbol	Name (Definition)	Page
G/H	Quotient space, quotient group (residue group) when H is normal subgroup	25
$\mathbb{Z}_k$	$\mathbb{Z}/k\mathbb{Z}$	26
$\mathbb{F}_p$	Finite field of order prime p ($\mathbb{Z}/p\mathbb{Z}$)	36
$\mathbb{F}_q$	Algebraic extension over $\mathbb{F}_p$ of degree n ($q = p^n$)	36
$\mathbb{F}_q^{(k)}$	k-dimensional subspace spanned by $e_1, \ldots, e_k$ over $\mathbb{F}_q$	277
$\mathbb{X}$	$\mathbb{F}_q$ or $\mathbb{Z}_d$	268

Table 2 Symbols for Lie algebras (see Tables 3.4, 3.5, and 3.6 for other Lie algebras)

Symbol	Name (Definition)	Page
$\mathfrak{g}_1 \oplus \mathfrak{g}_2$	Direct sum of $\mathfrak{g}_1$ and $\mathfrak{g}_2$	77
$\mathfrak{g}_1 \rtimes \mathfrak{g}_2$	Semi direct product of $\mathfrak{g}_1$ and $\mathfrak{g}_2$	90
$\mathfrak{g}_{\mathbb{C}}$	Complexification of $\mathfrak{g}$	77
$\mathfrak{c}(\mathfrak{g})$	Center of $\mathfrak{g}$	78
$\mathfrak{h}$	Cartan subalgebra	205
$\mathfrak{k}$	Maximal compact subalgebra	223
$\mathfrak{gl}(\mathcal{H})$	Set of linear maps from $\mathcal{H}$ to $\mathcal{H}$	37
$\mathfrak{u}(\mathcal{H})$	Set of skew-Hermitian matrices on $\mathcal{H}$	85
$\mathfrak{o}(V)$	Set of real alternative matrices on V	86

Table 3 Sets

Symbol	Name (Definition)	Page
$\hat{G}$	Set of irreducible unitary representations of G	41
$\hat{G}_{\mathfrak{f}}$	Set of finite-dimensional representations in $\hat{G}$	41
$\hat{G}([\{e^{i(g,g')}\}_{g,g'}])$	Set of irreducible projective unitary representations of factor system $[\{e^{i(g,g')}\}_{g,g'}] \in H(G : \mathrm{U}(1))$	48
$\hat{G}_{\mathfrak{f}}([\{e^{i(g,g')}\}_{g,g'}])$	Set of finite-dimensional projective representations in $\hat{G}([\{e^{i(g,g')}\}_{g,g'}])$	48
$\hat{\mathfrak{g}}$	Set of irreducible skew-Hermitian presentations of Lie algebra $\mathfrak{g}$	99
$\hat{\mathfrak{g}}_{\mathfrak{f}}$	Set of finite-dimensional representation in $\hat{\mathfrak{g}}$	99
$\mathcal{Y}^r$	Set of Young diagram of depth not greater than r	65
$\mathcal{Y}_n$	Set of Young diagram of size n	65
$\mathcal{Y}_n^r$	$\mathcal{Y}^r \cap \mathcal{Y}_n$	65
$Y_{U,r}(\boldsymbol{n})$	Set of Semistandard Young tableau corresponding to $\boldsymbol{n}$	65
$Y_S(\boldsymbol{n})$	Set of Standard Young tableau corresponding to $\boldsymbol{n}$	65
Φ	Root system	130, 201
$\Phi_{\mathfrak{g}}$	Root system of Lie algebra $\mathfrak{g}$	209
Δ	Fundamental system	130, 202
D	Unit circle	117

Table 4 Symbols for Representation

Symbol	Name (Definition)	Group (Lie algebra)	Representation space	Page
$\mathsf{f}_1 \oplus \mathsf{f}_2$	Direct sum representation	$G(\mathfrak{g})$	$\mathcal{H}_1 \oplus \mathcal{H}_2$	39, 86
$\mathsf{f}_1 \otimes \mathsf{f}_2$	Tensor product representation	$G(\mathfrak{g})$	$\mathcal{H}_1 \otimes \mathcal{H}_2$	39, 86
$\mathsf{f}_1 \bar{\otimes} \mathsf{f}_2$		$G_1 \times G_2$	$\mathcal{H}_1 \otimes \mathcal{H}_2$	44
ad	Adjoint representation	$G(\mathfrak{g})$	$\mathfrak{g}$	87
$\mathsf{f}_H \rtimes \mathsf{f}_K$	Semi direct product representation	$H \rtimes K$		50
$\mathsf{f}_H \tilde{\rtimes} \mathsf{f}_T$		$H \rtimes K$		50
$\mathsf{f}_H \rtimes \mathsf{f}_K \otimes \mathsf{f}'_K$		$H \rtimes K$		51
$\mathsf{f}_H \tilde{\rtimes} \mathsf{f}_T \otimes \mathsf{f}'_T$		$H \rtimes K$		51
$\bar{\mathsf{f}}$	Complex conjugate representation of f			53
$\tilde{\mathsf{f}}$		$\tilde{G}$		46, 98
ι		$\mathrm{SU}(2)$ $(\mathfrak{su}(2))$	$\mathbb{R}^3$	116
W	Heisenberg representation	$\mathbb{R}^2$, $H(2, \mathbb{R})$	$L^2(\mathbb{R})$	233
U		$\mathrm{U}(r)$	$L^2(\mathbb{R})$	238, 242
W^r	r-mode Heisenberg representation	$\mathbb{R}^{2r}$, $H(2r, \mathbb{R})$	$L^2(\mathbb{R}^r)$	241
S		$\mathrm{SU}(1,1)$ $(\mathfrak{su}(1,1))$	$L^2(\mathbb{R})$	250
S^2		$\mathrm{SU}(1,1)$ $(\mathfrak{su}(1,1))$	$L^2(\mathbb{R}^2)$	254
S^r		$\mathrm{S}p(2r, \mathbb{R})$	$L^2(\mathbb{R}^r)$	259
$\tilde{\mathsf{W}}_{\mathbb{Z}}$	Pre-discrete-Heisenberg representation	$\mathbb{Z}_d^2$	$\mathbb{C}^d$	264
$\mathsf{W}_{\mathbb{Z}}$	Discrete Heisenberg representation	$\mathbb{Z}_d^2$	$\mathbb{C}^d$	264
$\mathsf{W}_{\mathbb{Z},\mathrm{H}}$		$\mathrm{H}(2, \mathbb{Z}_d)$	$\mathbb{C}^d$	265
$\tilde{\mathsf{W}}_{\mathbb{F}}$	Pre-discrete-Heisenberg representation	$\mathbb{F}_q^2$	$\mathbb{C}^q$	267
$\mathsf{W}_{\mathbb{F}}$	Discrete Heisenberg representation	$\mathbb{F}_q^2$	$\mathbb{C}^q$	267
$\mathsf{W}^r_{\mathbb{X}}$	Discrete Heisenberg representation	$\mathbb{F}_q^{2r}$, $\mathbb{Z}_d^{2r}$	$\mathbb{C}^{d^r}$, $\mathbb{C}^{q^r}$	268
$\tilde{\mathsf{W}}^r_{\mathbb{X}}$	Pre-discrete-Heisenberg representation	$\mathbb{F}_q^{2r}$, $\mathbb{Z}_d^{2r}$	$\mathbb{C}^{d^r}$, $\mathbb{C}^{q^r}$	268
$\mathsf{W}_{\mathbb{Z},\mathrm{H}}$		$\mathrm{H}(2r, \mathbb{Z}_d)$	$\mathbb{C}^{d^r}$	270
$\mathsf{W}_{\mathbb{F},\mathrm{H}}$		$\mathrm{H}(2r, \mathbb{F}_q)$	$\mathbb{C}^{q^r}$	270
$\mathsf{S}^r_{\mathbb{X}}(g)$	Metaplectic representation	$\mathrm{S}p(2r, \mathbb{X})$	$\mathbb{C}^{d^r}$, $\mathbb{C}^{q^r}$	278
$\mathsf{V}^r_{\mathbb{X}}$		$\mathbb{X}^{2r} \rtimes \mathrm{S}p(2r, \mathbb{X})$	$\mathbb{C}^{d^r}$, $\mathbb{C}^{q^r}$	282

Table 5 Vector

Symbol	Name (Definition)	Page
$\vert X\rangle\rangle_{A,B}$	Vector on $\mathcal{H}_A \otimes \mathcal{H}_B$	9
$\vert\lambda;j\rangle$	CONS of $\mathcal{U}_\lambda(G)$ weight vector for SU(2)	41, 121
$\vert\lambda;\lambda\rangle$	Maximum weight vector	212, 225, 228
$\vert\lambda;j;j'\rangle$	$\vert\lambda;j\rangle \otimes \vert j'\rangle$	41
$\vert\lambda:\zeta\rangle$	Coherent vector	127
$\vert\lambda:\zeta\rangle$	Coherent vector	222, 227, 228
$\left\vert -\frac{1}{2}\boldsymbol{\upsilon}:\zeta\right\rangle$	Coherent vector	261
$\vert n\rangle$	Number vector	233
$\vert\hat{e}_l\rangle$	Dual computational base	58
$\vert\hat{e}_{\mathbb{Z}}(l)\rangle, \vert\hat{e}_{\mathbb{F}}(l)\rangle$	Dual computational base	265, 268

Table 6 Symbols for vector space

Symbol	Name (Definition)	Page
$\mathbb{C}[G]$	Group algebra	59
$L^2(G)$	Set of square integrable functions of G	61, 104, 107
$L^2(\hat{G})$		61, 104, 107
$L^2(G, inv)$	Set of square integrable functions of conjugated class	63
$L^2(G, inv, \mathbb{R})$	Set of real-valued functions in $L^2(G, inv)$	63
$L^2(\hat{G}[\mathcal{E}])$		64
$\mathcal{U}_\lambda(G), \mathcal{U}_\lambda$	Irreducible representation space of $\lambda \in \hat{G}$	41
$V \otimes \mathbb{C}$	Complexification of V with complex inner product	52
$V \bar{\otimes} \mathbb{C}$	Complexification of V with Hermitian inner product	52

Table 7 Symbols for matrices and operators

Symbol	Page
J	54
J_d	54
$E_{0,1}$	114
$K_{+,1}$	114
$K_{-,1}$	114
M_γ	114, 275
P_ζ	114, 275
Q_ζ	275
F_1^x	114
F_1^y	114
F_1^z	114
F_{-1}^x	116

(continued)

Table 7 (continued)

Symbol	Page
F^{y}_{-1}	116
F^{z}_{-1}	116
$E_{0,-1}$	117
$K_{+,-1}$	117
$K_{-,-1}$	117
W	118
$\Delta_{\rho}[\mathrm{Q}:\mathrm{P}]$	186
$\mathrm{X}_{\mathbb{Z}}$	278
$\mathrm{Z}_{\mathbb{Z}}$	278
$\mathrm{X}_{\mathbb{F}}(s)$	267
$\mathrm{Z}_{\mathbb{F}}(t)$	267
DFT	58
$F^{x}_{j,l}$	130
$F^{y}_{j,l}$	130
$F^{z}_{j,l}$	130
E_j	130
$K_{j,l;+}$	130
$K_{j,l;-}$	130
$F^{z,u}_{j}$	131
a	19, 186, 234
Q	19, 232
P	19, 232
N	234
$\mathrm{U}_{k,j}$	243

Table 8 Other symbols

Symbol	Name (Definition)	Page
$\overline{X}$	Complex conjugate matrix of X	9
X^T	Transposed matrix of X	9
$X^{\dagger}$	Transposed complex conjugate matrix of X	9
$\lvert G\rvert$	Order (number of elements) of G	23
$[g]$	Residue class with representative g	25
$\lvert G:H\rvert$	Index of H with respect to G	25
a^G	Conjugate class of a in G	26
$H \cdot K$	Product set of sets H and K	28
$\{T(k), \phi(k,k')\}$	Factor system	31
d_{λ}	Dimension or formal dimension of $U_{\lambda}(G)$	41, 102
$C^{\lambda''}_{\lambda,\lambda'}(G)$	Multiplicity	56

(continued)

Table 8 (continued)

Symbol	Name (Definition)	Page
χ_{f}	Character of representation f	41
χ_λ	Irreducible character	41
$\mathrm{supp}(\chi_{\mathsf{f}})$	Support of character χ_{f}	43
$\{\mathrm{e}^{i(g,g')}\}_{g,g'}$	Factor system	46
$H(\mathsf{f})$	Element of $H(G:\mathrm{U}(1))$ corresponding to projective representation f	46
$\lambda^* \in \hat{G}$	Label of complex conjugate representation of f_λ	53
$\mathcal{F}$	Fourier transform	63, 108
$\mathcal{F}^{-1}$	Inverse Fourier transform	63, 108
$(X,Y)_{\mathfrak{g}}$	Killing form	90
C_{f}	Casimir operator	94
μ_G	Normalized invariant measure over group G	102
μ_Θ	Normalized invariant measure over homogeneous space Θ	103
$\mu_n(r)$	Plancherel measure	67
$\mu_{\hat{G}[\mathcal{E}]}$	Plancherel measure	110
κ		118
$\mathcal{F}_{\mathrm{W}}$		255
$\omega_{\mathbb{Z}}$		264
$\tau_{\mathbb{Z}}$		264
$\omega_{\mathbb{F}}$		267
$\tau_{\mathbb{F}}$		267
$[m_1,\ldots,m_{r-1}]$	Dynkin index	130, 202
$(m_1,\ldots,m_{r-1})$	Young index	132
in		132
δ		137, 212