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The Monge-Ampère Equation

Second Edition

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Preface to the Second Edition

A considerable amount of material has been added to this edition. It contains two new chapters: Chapter 7 on the linearized Monge–Ampère equation and Chapter 8 on Hölder estimates for second derivatives of solutions to the Monge–Ampère equation. In addition, a set of 31 exercises is added to Chapter 1. The notes at the end of each chapter have been updated to reflect new developments since the publication of the first edition in 2001. Several misprints and errors from the first edition have been corrected, and more clarifications have been added.

Chapter 8 is written in collaboration with Qingbo Huang and Truyen Nguyen to whom I am also extremely grateful for numerous suggestions that improved the presentation.

I thank Farhan Abedin for carefully reading Chapters 1, 5, and 7 and for providing several suggestions that made some proofs more clear.

I hope this new edition will continue serving to stimulate research on the Monge–Ampère equation, its connections with several areas, and its applications.

Moorestown, NJ, USA
April 2016

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Preface to the First Edition

In recent years, the study of the Monge–Ampère equation has received considerable attention, and there have been many important advances. As a consequence, there is nowadays much interest in this equation and its applications. This volume tries to reflect these advances in an essentially self-contained systematic exposition of the theory of weak solutions, including recent regularity results by L. A. Caffarelli. The theory has a geometric flavor and uses some techniques from harmonic analysis such us covering lemmas and set decompositions. An overview of the contents of the book is as follows:

We shall be concerned with the Monge–Ampère equation, which for a smooth function u, is given by

$$\det D^2 u = f. \tag{0.0.1}$$

There is a notion of generalized or weak solution to (0.0.1): for u convex in a domain Ω, one can define a measure Mu in Ω such that if u is smooth, then Mu has density $\det D^2 u$. Therefore, u is a generalized solution of (0.0.1) if $Mu = f$. The notion of generalized solution is based on the notion of normal mapping, and in Chapter 1 we begin with these two concepts, introduced by A. D. Aleksandrov, and we describe their basic properties. The notion of viscosity solution is also considered and compared with that of generalized solution. We also introduce several maximum principles that are fundamental in the study of the Monge–Ampère operator. The Dirichlet problem for Monge–Ampère is then solved in the class of generalized solutions in Sections 1.5 and 1.6. Chapter 1 concludes with the concept of ellipsoid of minimum volume which is of particular importance in developing the theory of cross sections in Chapter 3.

In Chapter 2, we present the Harnack inequality of Krylov–Safonov for non-divergence elliptic operators in view of some ideas used to study the linearized Monge–Ampère equation. This illustrates these ideas in a case that is simpler than that of the linearized Monge–Ampère operator.

Chapter 3 presents the theory of cross sections of weak solutions to the Monge–Ampère equation, and we prove several geometric properties that are needed in the subsequent chapters. The cross sections of u are the level sets of the convex function u minus a supporting hyperplane. Of special importance is the doubling condition (3.1.1) for the measure Mu that permits us, from the characterization given in Theorem 3.3.5, to determine invariance properties for the shapes of cross sections that are valid under appropriate normalizations using ellipsoids of minimum volume. A typical situation is when the measure Mu satisfies

$$\lambda\,|E| \leq Mu(E) \leq \Lambda\,|E|, \tag{0.0.2}$$

for some positive constants λ, Λ and for all Borel subsets E of the convex domain Ω. The inequalities (0.0.2) resemble the uniform ellipticity condition for linear operators. The results proved in this chapter permit us to work with the cross sections as if they were Euclidean balls and to establish the covering lemmas needed later for the regularity theory in Chapters 4–6.

Chapter 4 concerns an application of the properties of the sections: a result of Jörgens–Calabi–Pogorelov–Cheng and Yau about the characterization of global solutions of $Mu = 1$.

Chapter 5 contains Caffarelli's $C^{1,\alpha}$ estimates for weak solutions. A fundamental geometric result is Theorem 5.2.1 about the extremal points of the set where a solution u equals a supporting hyperplane.

Finally, in Chapter 6, we present the $W^{2,p}$ estimates for the Monge–Ampère equation recently developed by Caffarelli and extend classical estimates of Pogorelov. The main result here is Theorem 6.4.2.

We have included bibliographical notes at the end of each chapter.

Acknowledgments

It is a pleasure to thank all the people who assisted me during the preparation of this book. I am particularly indebted to L. A. Caffarelli for inspiration, many discussions, and for his collaboration. I am very grateful to Qingbo Huang for innumerable enlightening discussions on most topics in this book, for many suggestions, and corrections, and for his collaboration. I am also very grateful to several friends and students for carefully reading various chapters of the manuscript: Shif Berhanu, Giuseppe Di Fazio, David Hartenstine, and Federico Tournier. They have made many helpful comments, suggestions and corrections that improved the presentation. I would especially like to thank L. C. Evans for his encouragement and suggestions.

This book encompasses the contents of a graduate course at Temple University, and some chapters have been used in short courses at the Università di Bologna, Universidad de Buenos Aires, and Universidad Autónoma de Madrid. I would like to thank these institutions and all my friends there for the kind hospitality and support.

The research connected with the results in this volume was supported in part by the National Science Foundation, and I wish to thank this institution for its support.

Cherry Hill, NJ, USA
September 2000

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Notation

Du denotes the gradient of the function u.

$D^2u(x)$ denotes the Hessian of the function u, i.e., $D^2u(x) = \left(\dfrac{\partial^2 u(x)}{\partial x_i \partial x_j}\right)$, $1 \leq i, j \leq n$.

$\Omega \subset \mathbb{R}^n$, $u : \Omega \to \mathbb{R}$ is convex if for all $0 \leq t \leq 1$ and any $x, y \in \Omega$ such that $tx + (1-t)y \in \Omega$ we have

$$u(tx + (1-t)y) \leq tu(x) + (1-t)u(y).$$

Given a set E, $\chi_E(x)$ denotes the characteristic function of E.

$|E|$ denotes the Lebesgue measure of the set E.

$B_R(x)$ denotes the Euclidean ball centered at x with radius R.

ω_n denotes the measure of the unit ball in $\mathbb{R}^n$.

$C(\Omega)$ denotes the class of real-valued functions that are continuous in Ω.

Given a positive integer k, $C^k(\Omega)$ denotes the class of real-valued functions that are continuously differentiable in Ω up to order k.

If E_k is a sequence of sets, then

$$E^* = \limsup_{n\to\infty} E_n = \cap_{n=1}^{\infty} \cup_{k=n}^{\infty} E_k; \qquad E_* = \liminf_{n\to\infty} E_n = \cup_{n=1}^{\infty} \cap_{k=n}^{\infty} E_k;$$

$$\chi_{E^*}(x) = \limsup_{n\to\infty} \chi_{E_n}(x); \qquad \chi_{E_*}(x) = \liminf_{n\to\infty} \chi_{E_n}(x).$$

The real-valued function u is harmonic in the open set $\Omega \subset \mathbb{R}^n$ if $u \in C^2(\Omega)$ and $\Delta u(x) = \sum_{i=1}^n \dfrac{\partial^2 u(x)}{\partial x_i^2} = 0$ in Ω.

If $\Omega \subset \mathbb{R}^n$ is a bounded and measurable set, the center of mass or barycenter of Ω is the point x^* defined by

$$x^* = \frac{1}{|\Omega|} \int_\Omega x \, dx.$$

If $A \subset B \subset \mathbb{R}^n$ and $\bar{A} \subset B$, then we write $A \Subset B$.
If $a, b \in \mathbb{R}$, then $a \vee b = \max\{a, b\}$.
If E is a set, then $\mathcal{P}(E)$ denotes the class of all subsets of E.
If $Q \subset \mathbb{R}^n$ is a cube and $\alpha > 0$, then αQ denotes the cube concentric with Q but with edge length equals α times the edge length of Q.

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