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Steven Roman

# An Introduction to the Language of Category Theory

Birkhäuser

**Steven Roman**
California State University, Fullerton
Irvine, CA, USA

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*To Donna*

# Preface

The purpose of this little book is to provide an introduction to the *basic concepts* of category theory. It is intended for the graduate student, advanced undergraduate student, nonspecialist mathematician or scientist working in a need-to-know area. The treatment is abstract in nature, with examples drawn mainly from abstract algebra. Although there are no formal prerequisites for this book, a basic knowledge of elementary abstract algebra would be of considerable help, especially in dealing with the exercises.

Category theory is a relatively young subject, founded in the mid 1940s, with the lofty goals of *unification*, *clarification* and *efficiency* in mathematics. Indeed, Saunders MacLane, one of the founding fathers of category theory (along with Samuel Eilenberg), says in the first sentence of his book *Categories for the Working Mathematician*: "Category theory starts with the observation that many properties of mathematical systems can be unified and simplified by a presentation with diagrams of arrows." Of course, unification and simplification are common themes throughout mathematics.

To illustrate these concepts, consider three sets with a binary operation:

1) the set $\mathbb{R}^*$ of nonzero real numbers under multiplication
2) the set $\mathcal{M}(n, k)$ of $n \times k$ matrices over the complex numbers under addition and
3) the set $\mathcal{B}$ of bijections of the integers under composition.

Now, very few mathematicians would take the time to prove that inverse elements are unique in each of these cases—They would simply note that each of these is an example of a *group* and prove in one quick line that the inverse of *any* element in *any* group is unique, to wit, if $\alpha$ and $\beta$ are inverses for the group element $a$, then

$$\alpha = \alpha 1 = \alpha(\alpha\beta) = (\alpha a)\beta = 1\beta = \beta$$

This at once *clarifies* the role of uniqueness of inverses by showing that this property has *nothing whatever to do with real numbers, matrices or bijections*. It has to do only with associativity and the identity property. This also *unifies* the concept of uniqueness of inverses because it shows that uniqueness of inverses in each of these three cases is really a single concept. Finally, it makes life more *efficient* for mathematicians because they can prove uniqueness of inverses for *all* groups *in one fell swoop*, as it were.

Category theory attempts to do the same for *all* of mathematics (well, perhaps not *all*) as group theory does for the case described above.

But there is a problem. It has been my experience that most students of mathematics and the sciences (and even some mathematicians) find category theory to be very challenging indeed, primarily due to its extremely abstract nature. We must remember that the vast majority of students are *not* seeking to be category theorists—They are seeking a "modest" understanding of the *basic concepts* of category theory so that they can apply these ideas to their chosen area of specialty. This book attempts to supply this understanding in as gentle a manner as possible.

We envision this book as being used as independent reading or as a supplementary text for graduate courses in other areas. It could also be used as the textbook for either a short course or a leisurely one-quarter course in category theory.

## The Five Basic Concepts of Category Theory

It can be said that there are five *basic* concepts in category theory, namely,

- *Categories*
- *Functors*
- *Natural transformations*
- *Universality*
- *Adjoints*

Some would argue that each of these concepts was "invented" or "discovered" in order to produce the next concept in this list. For example, Saunders MacLane himself is reported to have said: "I did not invent category theory to talk about functors. I invented it to talk about natural transformations."

Whether this be true or not, many students of mathematics are finding that the language of category theory is popping up in many of their classes in abstract algebra, lattice theory, number theory, differential geometry, algebraic topology and more. Also, category theory has become an important topic of study for many computer scientists and even for some mathematical physicists. Hopefully, this book will fill a need for those who require an understanding of the basic concepts of the subject. If the need or desire should arise, one can then turn to more lengthy and advanced treatments of the subject.

## A Word About Definitions

To my mind, there are two types of definitions. *Standard definitions* are, well, standard. They are intended to be in common usage and last through the ages. However, after about 40 years of teaching and the writing of about 40 books, I have come to believe in the virtue of *nonstandard, temporary, ad hoc definitions* that are primarily intended for pedagogical purposes, although one can hope that some ad hoc definitions turn out to be so useful that they eventually become a standard part of the subject matter.

Let me illustrate a nonstandard definition. One of the most important (some would say *the* most important) concepts in category theory is that of an *adjoint*. There are left adjoints and right adjoints, but the two concepts come together in something called an *adjunction*.

Now, there are many approaches to discussing adjoints and adjunctions. In my experience, adjunctions are usually just defined without much preliminary leg work. However, one of the goals of this book is to make the more difficult concepts, such as that of an adjunction a bit more palatable by "sneaking up" on them, as it were. In order to do this with adjunctions, we gently transition through the following concepts,

$$\text{initial objects in comma categories} \to \text{universality} \to \text{naturalness} \to \text{binaturalness (adjunctions)}$$

During this transition process, we will find it extremely useful to use certain maps that, to my knowledge, do not have a specific name. So this is the perfect place to introduce a nonstandard definition, which in this case is the *mediating morphism map*.

The only downside to making nonstandard definitions is that they are not going to be recognized outside the context of this book and therefore must be used very circumspectly. But I think that is a small price to pay if they help the learning process.

That said, I will use nonstandard definitions only as often as I feel absolutely necessary and will try to identify them as such upon first use, either by the term "nonstandard" or by a phrase such as "we will refer to . . .".

## Thanks

I would like to thank my students Phong Le, Sunil Chetty, Timothy Choi, Josh Chan, Tim Tran and Zachary Faubion, who attended my lectures on a much expanded version of this book and offered many helpful suggestions.

**Steven Roman**

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