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Geometry and Quantization of Moduli Spaces

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Foreword

This book is based on four advanced courses given during a semester on *The Geometry and Quantization of Moduli Spaces*, held at the Centre de Recerca Matemàtica (CRM) in Bellaterra, Barcelona, from March to June 2012. Besides their important role in many areas of mathematics, in the last decades moduli spaces turned out to be crucial for the understanding of phenomena in high energy physics and, as such, have led to an interplay between mathematics and physics that has been amazingly fruitful for both sciences. We hope the reader will appreciate and enjoy the beauty of this interaction in this volume.

The courses were devoted to several topics that are experiencing extraordinary growth within the broad research area of moduli spaces, and reflected a recurrent thread in the research semester, namely the moduli space of local systems on a Riemann surface, both from classical and quantum perspectives.

Chapter 1, written by V.V. Fock (IRMA, Strasbourg, France) and A. Marshakov (ITEP, Moscow, Russia) and entitled *"Loop Groups, Clusters, Dimers and Integrable Systems"*, is framed in the context of cluster integrable systems, more precisely, the class of integrable systems constructed by A.B. Goncharov and R. Kenyon using the theory of dimer models in the statistical physics of bipartite graphs on a 2-dimensional torus. This research article describes an isomorphism between these integrable systems and standard ones constructed on appropriate affine Poisson–Lie groups, hence providing a new construction of the former and new viewpoints on the latter. The chapter includes very interesting examples, and points out at exciting generalizations.

Chapter 2, written by F. Schaffhauser (Universidad de los Andes, Bogotá, Colombia), and entitled *"Lectures on Klein Surfaces and Their Fundamental Group"*, is an expository survey of the fundamental group of a Klein surface. It includes an introduction to Klein surfaces and real algebraic curves, a careful account of the fundamental group of a Klein surface, and a discussion of linear and unitary representations of the fundamental group of a Klein surface, and the unitary representation varieties. Many definitions and results are motivated or illustrated with simple but descriptive examples, and each section has interesting exercises that the reader can use to reinforce the material.

Next is Chapter 3, written by C. Teleman (University of Oxford, Oxford, UK) and entitled *"Five Lectures on Topological Field Theory"*. Beginning from the classic definition of topological quantum field theories (TQFTs) systematized by M.F. Atiyah, it goes on through different examples to motivate the additional structure which these theories are now known to enjoy in many cases, as predicted by the so-called *cobordism hypothesis* proved recently by J. Lurie. Rather than giving a fully detailed definition of extended TQFTs, that the reader will find in Lurie's paper, the author explains, using well chosen examples, the substantial aspects of the notion, as well as the difficulties to prove its consistency and the cobordism hypothesis. This makes this text, in our opinion, an excellent companion to Lurie's paper.

The fourth and last is Chapter 4, written by R.A. Wentworth (University of Maryland, College Park, USA) and entitled *"Higgs Bundles and Local Systems on Riemann Surfaces"*. Higgs bundles, introduced by N. Hitchin thirty years ago, have proved to be of central importance in different areas, such as low dimensional topology, algebraic geometry, mathematical physics, or even number theory (e.g., in the proof of the fundamental lemma in the Langlands program by B.C. Ngô a few years ago). This chapter gives a pedagogical introduction to the notion restricted to Riemann surfaces (which was the original context in Hitchin's papers). It gives an almost self-contained proof of the correspondence between the moduli spaces of Higgs bundles and local systems, and then applies it to study the oper moduli space, an object playing a crucial role in the recent developments on the geometric Langlands program.

We would like to thank the authors of the present book for their great work, both in preparing a very interesting set of lectures and for writing the present notes. We are also grateful to the referees who helped us uncompromisingly.

The semester on *The Geometry and Quantization of Moduli Spaces* was made possible by the generous financial support of different institutions. We would like to thank for this reason the Centre de Recerca Matemàtica, the European Research Foundation through the research network ITGP (Interactions between Topology, Geometry and Physics), the IMUB (Institut de Matemàtiques de la Universitat de Barcelona) and the SCM (Societat Catalana de Matemàtiques). Finally, we would like to thank the staff of the CRM for their efficiency and help during the preparation and running of the program.

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