

Graduate Texts in Physics

Series editors

Jean-Marc Di Meglio, Paris, France

William T. Rhodes, Boca Raton, USA

Susan Scott, Acton, Australia

Martin Stutzmann, Garching, Germany

Andreas Wipf, Jena, Germany

Graduate Texts in Physics

Graduate Texts in Physics publishes core learning/teaching material for graduate- and advanced-level undergraduate courses on topics of current and emerging fields within physics, both pure and applied. These textbooks serve students at the MS- or PhD-level and their instructors as comprehensive sources of principles, definitions, derivations, experiments and applications (as relevant) for their mastery and teaching, respectively. International in scope and relevance, the textbooks correspond to course syllabi sufficiently to serve as required reading. Their didactic style, comprehensiveness and coverage of fundamental material also make them suitable as introductions or references for scientists entering, or requiring timely knowledge of, a research field.

More information about this series at <http://www.springer.com/series/8431>

Simon Širca

Probability for Physicists

 Springer

Simon Širca
Faculty of Mathematics and Physics
University of Ljubljana
Ljubljana
Slovenia

ISSN 1868-4513

ISSN 1868-4521 (electronic)

Graduate Texts in Physics

ISBN 978-3-319-31609-3

ISBN 978-3-319-31611-6 (eBook)

DOI 10.1007/978-3-319-31611-6

Library of Congress Control Number: 2016937517

© Springer International Publishing Switzerland 2016

This work is subject to copyright. All rights are reserved by the Publisher, whether the whole or part of the material is concerned, specifically the rights of translation, reprinting, reuse of illustrations, recitation, broadcasting, reproduction on microfilms or in any other physical way, and transmission or information storage and retrieval, electronic adaptation, computer software, or by similar or dissimilar methodology now known or hereafter developed.

The use of general descriptive names, registered names, trademarks, service marks, etc. in this publication does not imply, even in the absence of a specific statement, that such names are exempt from the relevant protective laws and regulations and therefore free for general use.

The publisher, the authors and the editors are safe to assume that the advice and information in this book are believed to be true and accurate at the date of publication. Neither the publisher nor the authors or the editors give a warranty, express or implied, with respect to the material contained herein or for any errors or omissions that may have been made.

Printed on acid-free paper

This Springer imprint is published by Springer Nature

The registered company is Springer International Publishing AG Switzerland

Preface

University-level introductory books on probability and statistics tend to be long—too long for the attention span and immediate horizon of a typical physics student who might wish to absorb the necessary topics in a swift, direct, involving manner, relying on her existing knowledge and physics intuition rather than asking to be taken through the content at a slow and perhaps over-systematic pace.

In contrast, this book attempts to deliver a concise, lively, intuitive introduction to probability and statistics for undergraduate and graduate students of physics and other natural sciences. Conceived primarily as a text for the second-year course on *Probability in Physics* at the Department of Physics, Faculty of Mathematics and Physics, University of Ljubljana, it has been designed to be as relieved of unnecessary mathematical ballast as possible, yet never to be mathematically imprecise. At the same time, it is hoped to be colorful and captivating: to this end, I have strived to avoid endless, dry prototypes with tossing coins, throwing dice and births of girls and boys, and replace them wherever possible by physics-motivated examples, always in the faith that the reader is already familiar with “at least something”. The book also tries to fill a few common gaps and resurrect some content that seems to be disappearing irretrievably from the modern, Bologna-style curricula. Typical witnesses of such efforts are the sections on extreme-value distributions, linear regression by using singular-value decomposition, and the maximum-likelihood method.

The book consists of four parts. In the first part (Chaps. 1–6) we discuss the fundamentals of probability and probability distributions. The second part (Chaps. 7–10) is devoted to statistics, that is, the determination of distribution parameters based on samples. Chapters 11–14 of the third part are “applied”, as they are the place to reap what has been sown in the first two parts and they invite the reader to a more concrete, computer-based engagement. As such, these chapters lack the concluding exercise sections, but incorporate extended examples in the main text. The fourth part consists of appendices. Optional contents are denoted by asterisks \star . Without them, the book is tailored to a compact one-semester course;

with them included, it can perhaps serve as a vantage point for a two-semester agenda.

The story-telling and the style are mine; regarding all other issues and doubts I have gladly obeyed the advice of both benevolent, though merciless reviewers, Dr. Martin Horvat and Dr. Gregor Šega. Martin is a treasure-trove of knowledge on an incredible variety of problems in mathematical physics, and in particular of *answers* to these problems. He does not terminate the discussions with the elusive “The solution exists!”, but rather with a fully functional, tested and documented computer code. His ad hoc products saved me many hours of work. Gregor has shaken my conviction that a partly loose, intuitive notation could be reader-friendly. He helped to furnish the text with an appropriate measure of mathematical rigor, so that I could ultimately run with the physics hare and hunt with the mathematics hounds. I am grateful to them for reading the manuscript so attentively. I would also like to thank my student Mr. Peter Ferjančič for leading the problem-solving classes for two years and for suggesting and solving Problem 5.6.3.

I wish to express my gratitude to Professor Claus Ascheron, Senior Editor at Springer, for his effort in preparation and advancement of this book, as well as to Viradasarani Natarajan and his team for its production at Scientific Publishing Services. <http://pp.books.fmf.uni-lj.si>

Ljubljana

Simon Širca

Contents

Part I Fundamentals of Probability and Probability Distributions

1	Basic Terminology	3
1.1	Random Experiments and Events	3
1.2	Basic Combinatorics	6
1.2.1	Variations and Permutations	6
1.2.2	Combinations Without Repetition	7
1.2.3	Combinations with Repetition	8
1.3	Properties of Probability	8
1.4	Conditional Probability	11
1.4.1	Independent Events	14
1.4.2	Bayes Formula	16
1.5	Problems	18
1.5.1	Boltzmann, Bose–Einstein and Fermi–Dirac Distributions	18
1.5.2	Blood Types	19
1.5.3	Independence of Events in Particle Detection	21
1.5.4	Searching for the Lost Plane	22
1.5.5	The Monty Hall Problem ★	22
1.5.6	Bayes Formula in Medical Diagnostics	25
1.5.7	One-Dimensional Random Walk ★	27
	References	29
2	Probability Distributions	31
2.1	Dirac Delta	31
2.1.1	Composition of the Dirac Delta with a Function	33
2.2	Heaviside Function	35
2.3	Discrete and Continuous Distributions	36
2.4	Random Variables	37
2.5	One-Dimensional Discrete Distributions	37
2.6	One-Dimensional Continuous Distributions	39

2.7	Transformation of Random Variables	41
2.7.1	What If the Inverse of $y = h(x)$ Is Not Unique?	44
2.8	Two-Dimensional Discrete Distributions	45
2.9	Two-Dimensional Continuous Distributions	47
2.10	Transformation of Variables in Two and More Dimensions	50
2.11	Problems.	56
2.11.1	Black-Body Radiation.	56
2.11.2	Energy Losses of Particles in a Planar Detector	57
2.11.3	Computing Marginal Probability Densities from a Joint Density.	58
2.11.4	Independence of Random Variables in Two Dimensions	60
2.11.5	Transformation of Variables in Two Dimensions	61
2.11.6	Distribution of Maximal and Minimal Values	63
	References	64
3	Special Continuous Probability Distributions.	65
3.1	Uniform Distribution	65
3.2	Exponential Distribution	67
3.2.1	Is the Decay of Unstable States Truly Exponential?	70
3.3	Normal (Gauss) Distribution	70
3.3.1	Standardized Normal Distribution.	71
3.3.2	Measure of Peak Separation	73
3.4	Maxwell Distribution	74
3.5	Pareto Distribution	75
3.5.1	Estimating the Maximum x in the Sample	77
3.6	Cauchy Distribution	77
3.7	The χ^2 distribution	79
3.8	Student's Distribution.	79
3.9	F distribution	80
3.10	Problems.	80
3.10.1	In-Flight Decay of Neutral Pions	80
3.10.2	Product of Uniformly Distributed Variables.	83
3.10.3	Joint Distribution of Exponential Variables	84
3.10.4	Integral of Maxwell Distribution over Finite Range	85
3.10.5	Decay of Unstable States and the Hyper-exponential Distribution.	86
3.10.6	Nuclear Decay Chains and the Hypo-exponential Distribution.	89
	References	91
4	Expected Values	93
4.1	Expected (Average, Mean) Value.	93
4.2	Median.	95
4.3	Quantiles	96

- 4.4 Expected Values of Functions of Random Variables. 98
 - 4.4.1 Probability Densities in Quantum Mechanics. 99
- 4.5 Variance and Effective Deviation. 100
- 4.6 Complex Random Variables. 101
- 4.7 Moments. 102
 - 4.7.1 Moments of the Cauchy Distribution. 105
- 4.8 Two- and d -dimensional Generalizations. 106
 - 4.8.1 Multivariate Normal Distribution. 110
 - 4.8.2 Correlation Does Not Imply Causality. 111
- 4.9 Propagation of Errors. 111
 - 4.9.1 Multiple Functions and Transformation
of the Covariance Matrix. 113
- 4.10 Problems. 115
 - 4.10.1 Expected Device Failure Time. 115
 - 4.10.2 Covariance of Continuous Random Variables. 116
 - 4.10.3 Conditional Expected Values of Two-Dimensional
Distributions. 117
 - 4.10.4 Expected Values of Hyper- and Hypo-exponential
Variables. 117
 - 4.10.5 Gaussian Noise in an Electric Circuit. 119
 - 4.10.6 Error Propagation in a Measurement
of the Momentum Vector \star 120
- References. 121
- 5 Special Discrete Probability Distributions. 123**
 - 5.1 Binomial Distribution. 123
 - 5.1.1 Expected Value and Variance. 126
 - 5.2 Multinomial Distribution. 128
 - 5.3 Negative Binomial (Pascal) Distribution. 129
 - 5.3.1 Negative Binomial Distribution of Order k 129
 - 5.4 Normal Approximation of the Binomial Distribution. 130
 - 5.5 Poisson Distribution. 132
 - 5.6 Problems. 135
 - 5.6.1 Detection Efficiency. 135
 - 5.6.2 The Newsboy Problem \star 136
 - 5.6.3 Time to Critical Error. 138
 - 5.6.4 Counting Events with an Inefficient Detector. 140
 - 5.6.5 Influence of Primary Ionization on Spatial
Resolution \star 140
- References. 142
- 6 Stable Distributions and Random Walks. 143**
 - 6.1 Convolution of Continuous Distributions. 143
 - 6.1.1 The Effect of Convolution on Distribution
Moments. 146

6.2	Convolution of Discrete Distributions	147
6.3	Central Limit Theorem	149
6.3.1	Proof of the Central Limit Theorem	150
6.4	Stable Distributions ★	153
6.5	Generalized Central Limit Theorem ★	155
6.6	Extreme-Value Distributions ★	156
6.6.1	Fisher–Tippett–Gnedenko Theorem	158
6.6.2	Return Values and Return Periods	159
6.6.3	Asymptotics of Minimal Values.	161
6.7	Discrete-Time Random Walks ★	162
6.7.1	Asymptotics	163
6.8	Continuous-Time Random Walks ★	165
6.9	Problems.	167
6.9.1	Convolutions with the Normal Distribution	167
6.9.2	Spectral Line Width	168
6.9.3	Random Structure of Polymer Molecules	169
6.9.4	Scattering of Thermal Neutrons in Lead	171
6.9.5	Distribution of Extreme Values of Normal Variables ★	172
	References	174

Part II Determination of Distribution Parameters

7	Statistical Inference from Samples	177
7.1	Statistics and Estimators	178
7.1.1	Sample Mean and Sample Variance	179
7.2	Three Important Sample Distributions.	184
7.2.1	Sample Distribution of Sums and Differences	184
7.2.2	Sample Distribution of Variances	185
7.2.3	Sample Distribution of Variance Ratios.	186
7.3	Confidence Intervals.	188
7.3.1	Confidence Interval for Sample Mean	188
7.3.2	Confidence Interval for Sample Variance	191
7.3.3	Confidence Region for Sample Mean and Variance	191
7.4	Outliers and Robust Measures of Mean and Variance	192
7.4.1	Chasing Outliers	193
7.4.2	Distribution of Sample Median (and Sample Quantiles).	194
7.5	Sample Correlation.	195
7.5.1	Linear (Pearson) Correlation	195
7.5.2	Non-parametric (Spearman) Correlation	196

- 7.6 Problems. 198
 - 7.6.1 Estimator of Third Moment. 198
 - 7.6.2 Unbiasedness of Poisson Variable Estimators. 199
 - 7.6.3 Concentration of Mercury in Fish. 199
 - 7.6.4 Dosage of Active Ingredient 201
- References 201
- 8 Maximum-Likelihood Method 203**
 - 8.1 Likelihood Function 203
 - 8.2 Principle of Maximum Likelihood 204
 - 8.3 Variance of Estimator. 206
 - 8.3.1 Limit of Large Samples 207
 - 8.4 Efficiency of Estimator 209
 - 8.5 Likelihood Intervals 212
 - 8.6 Simultaneous Determination of Multiple Parameters 214
 - 8.6.1 General Method for Arbitrary (Small or Large) Samples 214
 - 8.6.2 Asymptotic Method (Large Samples) 215
 - 8.7 Likelihood Regions 217
 - 8.7.1 Alternative Likelihood Regions 218
 - 8.8 Problems. 219
 - 8.8.1 Lifetime of Particles in Finite Detector 219
 - 8.8.2 Device Failure Due to Corrosion 221
 - 8.8.3 Distribution of Extreme Rainfall 222
 - 8.8.4 Tensile Strength of Glass Fibers. 224
- References 224
- 9 Method of Least Squares 227**
 - 9.1 Linear Regression 228
 - 9.1.1 Fitting a Polynomial, Known Uncertainties 230
 - 9.1.2 Fitting Observations with Unknown Uncertainties 232
 - 9.1.3 Confidence Intervals for Optimal Parameters 235
 - 9.1.4 How “Good” Is the Fit? 236
 - 9.1.5 Regression with Orthogonal Polynomials ★ 236
 - 9.1.6 Fitting a Straight Line. 237
 - 9.1.7 Fitting a Straight Line with Uncertainties in both Coordinates. 240
 - 9.1.8 Fitting a Constant. 240
 - 9.1.9 Are We Allowed to Simply Discard Some Data? 242
 - 9.2 Linear Regression for Binned Data. 242
 - 9.3 Linear Regression with Constraints 245
 - 9.4 General Linear Regression by Singular-Value Decomposition ★ 248
 - 9.5 Robust Linear Regression 249
 - 9.6 Non-linear Regression 250

- 9.7 Problems. 253
 - 9.7.1 Two Gaussians on Exponential Background 253
 - 9.7.2 Time Dependence of the Pressure Gradient 254
 - 9.7.3 Thermal Expansion of Copper 255
 - 9.7.4 Electron Mobility in Semiconductor 255
 - 9.7.5 Quantum Defects in Iodine Atoms 256
 - 9.7.6 Magnetization in Superconductor 257
- References 257

10 Statistical Tests: Verifying Hypotheses 259

- 10.1 Basic Concepts 259
- 10.2 Parametric Tests for Normal Variables 264
 - 10.2.1 Test of Sample Mean 264
 - 10.2.2 Test of Sample Variance. 265
 - 10.2.3 Comparison of Two Sample Means, $\sigma_X^2 = \sigma_Y^2$ 265
 - 10.2.4 Comparison of Two Sample Means, $\sigma_X^2 \neq \sigma_Y^2$ 266
 - 10.2.5 Comparison of Two Sample Variances 267
- 10.3 Pearson’s χ^2 Test 269
 - 10.3.1 Comparing Two Sets of Binned Data 271
- 10.4 Kolmogorov–Smirnov Test 271
 - 10.4.1 Comparison of Two Samples. 274
 - 10.4.2 Other Tests Based on Empirical Distribution Functions 275
- 10.5 Problems. 276
 - 10.5.1 Test of Mean Decay Time. 276
 - 10.5.2 Pearson’s Test for Two Histogrammed Samples. 278
 - 10.5.3 Flu Medicine. 279
 - 10.5.4 Exam Grades. 279
- References 280

Part III Special Applications of Probability

11 Entropy and Information ★ 283

- 11.1 Measures of Information and Entropy. 283
 - 11.1.1 Entropy of Infinite Discrete Probability Distribution. 285
 - 11.1.2 Entropy of a Continuous Probability Distribution 286
 - 11.1.3 Kullback–Leibler Distance. 287
- 11.2 Principle of Maximum Entropy 288
- 11.3 Discrete Distributions with Maximum Entropy. 289
 - 11.3.1 Lagrange Formalism for Discrete Distributions 289
 - 11.3.2 Distribution with Prescribed Mean and Maximum Entropy. 291
 - 11.3.3 Maxwell–Boltzmann Distribution 292

11.3.4	Relation Between Information and Thermodynamic Entropy	294
11.3.5	Bose–Einstein Distribution	295
11.3.6	Fermi–Dirac Distribution.	296
11.4	Continuous Distributions with Maximum Entropy	297
11.5	Maximum-Entropy Spectral Analysis	298
11.5.1	Calculating the Lagrange Multipliers	300
11.5.2	Estimating the Spectrum	302
	References	304
12	Markov Processes ★	307
12.1	Discrete-Time (Classical) Markov Chains	308
12.1.1	Long-Time Characteristics of Markov Chains	309
12.2	Continuous-Time Markov Processes	313
12.2.1	Markov Propagator and Its Moments	314
12.2.2	Time Evolution of the Moments	316
12.2.3	Wiener Process	317
12.2.4	Ornstein–Uhlenbeck Process	318
	References	323
13	The Monte–Carlo Method	325
13.1	Historical Introduction and Basic Idea	325
13.2	Numerical Integration.	328
13.2.1	Advantage of Monte–Carlo Methods over Quadrature Formulas.	332
13.3	Variance Reduction	333
13.3.1	Importance Sampling	333
13.3.2	The Monte–Carlo Method with Quasi-Random Sequences.	337
13.4	Markov-Chain Monte Carlo ★	339
13.4.1	Metropolis–Hastings Algorithm	339
	References	345
14	Stochastic Population Modeling	347
14.1	Modeling Births.	347
14.2	Modeling Deaths	348
14.3	Modeling Births and Deaths	351
14.3.1	Equilibrium State	352
14.3.2	General Solution in the Case $\lambda_n = N\lambda$, $\mu_n = N\mu$, $\lambda \neq \mu$	353
14.3.3	General Solution in the Case $\lambda_n = N\lambda$, $\mu_n = N\mu$, $\lambda = \mu$	353
14.3.4	Extinction Probability	353
14.3.5	Moments of the Distribution $P(t)$ in the Case $\lambda_n = n\lambda$, $\mu_n = n\mu$	354

14.4 Concluding Example: Rabbits and Foxes	357
References	359
Appendix A: Probability as Measure ★.	361
Appendix B: Generating and Characteristic Functions ★.	365
Appendix C: Random Number Generators.	381
Appendix D: Tables of Distribution Quantiles	395
Index	409