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Michel Grabisch

# Set Functions, Games and Capacities in Decision Making

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# Foreword

The book by Michel Grabisch is about a fascinating mathematical object that has received different names and has been studied by different communities: set functions, capacities, pseudo-Boolean functions, and cooperative games, to mention just a few. Results on these objects were often proven several times, often under slightly different forms, in different communities. The book has two main parts.

The first one is devoted to a detailed presentation of this mathematical object and its main properties. In particular, Michel gives a detailed presentation of the notion of core and of the integrals based on nonadditive measures. In this first part, the learning curve is steep, but the reward is quite worth the effort. Many results scattered in the literature are arranged and proved here in a unified framework. I have no doubt that this first part will serve as a reference text for all persons working in the field.

The second part deals with applications of this mathematical object. Three of them are emphasized: decision-making under risk and uncertainty, multiple criteria decision-making, and belief and plausibility measures in the spirit of Dempster and Shafer. A reader interested in these areas of application can directly start reading the book with one of these chapters. The style of exposition is such that the reader is given many useful hints and precise references to the first part of the book. It will be most useful to anyone willing to use the tools and concepts in his/her own research.

The book is quite rich and very pleasant to read. The technical parts are well organized, and difficult points are always illustrated by figures and examples. The application parts are lucidly written and should be accessible to many readers.

This should not be surprising. Michel has been a major figure in the area since nearly 30 years. He has fully succeeded in transforming his deep knowledge of the field into a very rich and quite readable text.

CNRS and Université Paris Dauphine — Denis Bouyssou
Paris, France
March 2016

# Preface

Set functions are mappings that assign to subsets of a universal set a real number and appear in many fields of mathematics (pure and applied) and computer sciences: combinatorics, measure and integration theory, combinatorial optimization, reliability, graph theory, cryptography, operations research in general, decision theory, game theory, etc. While additive nonnegative set functions (called measures) have been studied in depth and from a long time in measure theory, nonadditive set functions received less attention and only from, roughly speaking, the last 50 or 60 years. As the foregoing list shows, (nonadditive) set functions appeared in many different fields, under different names, and most often in an independent way. As far as possible, we have tried to give the historical origins of the concepts presented in this monograph. One of the most prominent seminal work is undoubtedly the one of Gustave Choquet, who proposed in 1953 the concept of *capacity* (monotone set function). Largely ignored during several decades, reinvented in 1974 by Michio Sugeno under the name *fuzzy measures*, capacities have become a central tool in all areas of decision-making, in particular, owing to the pioneering work of David Schmeidler in 1986. At the same time of Choquet's work on capacities, Lloyd Shapley studied another type of set functions, namely, *transferable utility games in characteristic form* (which we call here "game" for brevity), introduced by John von Neumann and Oskar Morgenstern, giving rise to what is known today as cooperative game theory. Submodular games happened to be of particular importance in combinatorial optimization through the work of Jack Edmonds, and many results concerning this class of games have been shown independently in both domains. Lastly, set functions, viewed as real-valued functions on the vertices of the unit hypercube, have been studied in the 1960s by Peter Hammer under the name *pseudo-Boolean functions* and constitute an important tool in operations research. This brief historical perspective, first, explains the title of this book, which is a compromise between the laconic "set functions" and the verbose "set functions, capacities, games, and pseudo-Boolean functions in decision-making, game theory, and operations research," and, second, gives an idea about the difficulty to have a clear view about what is known on set functions, from a mathematical point of view. As it is common in sciences and especially in our times where specialization reigns (a feature that will certainly worsen with the mania of evaluation and bibliometrics), scientific communities work independently and ignore that some of them share the same (mathematical) concerns. This book is an attempt to give a unified view of set functions and their avatars in the above-mentioned fields, mainly decision-making, cooperative game theory, and operations research, focusing on mathematical properties and presented in a way which is free of any particular applicative context. I mainly work with a finite universal set, first because most of the application fields concerned here consider finite sets (with the exception of decision under uncertainty and risk) and second because infinite sets require radically different mathematical tools, in the present case, close to those of measure theory (needless to say, a rigorous treatment of this would require a second volume, at least as thick as this one, a task which is probably beyond my capabilities!). The seven chapters are divided into three parts:

- Chapter 1 (introductive) establishes the notation and gathers the main mathematical ingredients which are necessary to understand the book.
- Chapters 2–4 (fundamental), which represent almost ⅔ of the book, form the mathematical core of the book. They give the mathematical properties of set functions, games, and capacities (Chap. 2), of the core of games, that is, the set of measures dominating a given game (Chap. 3), and of the various integrals defined w.r.t. games and capacities, mainly the Choquet and Sugeno integrals. At very few exceptions, all proofs are given.
- Chapters 5–7 (applicative) are devoted to applied domains: decision under risk and uncertainty (Chap. 5), decision with multiple criteria (Chap. 6), and Dempster-Shafer and possibility theory (Chap. 7). Clearly, each of these topics would have required a whole book, and at least for the two first ones, already many books are available on the topic. My philosophy was therefore different from the other chapters, and I tried to emphasize there the use of capacities. For these reasons, few proofs are given, but those given concern results which are either new or difficult to find in the literature. Chapter 7 is a bit in the spirit of the fundamental chapters, and therefore almost all proofs are provided. This is because, unlike the two chapters on decision-making, the topic is not so well known and still lacks comprehensive monographs.

The applicative chapters can be read independently from one another. It is also possible to read them without having studied in depth the fundamental chapters, because necessary concepts and results from these chapters are always clearly indicated and referenced.

The idea of writing this book germinated in my mind many years ago while teaching a course on capacities and Choquet integral applied to decision-making to second year master's students. I started the writing in 2012 and realized that it will take much time and go far beyond the initial project, when I saw that the first three pages of my handwritten lecture notes developed little by little into the hundred pages of Chap. 2. Anyway, the trip through the world of set functions was long, exhausting, but fascinating. Such a trip would have never existed if Prof. Michio Sugeno would not have permitted me to stay 1 year in his laboratory in 1989–1990, where I discovered his work on fuzzy measures and fuzzy integrals. I owe him to have introduced me to this beautiful world, which has become my main topic of research, and for this, I would like to express my most sincere gratitude to him. Many thanks are due also to his colleague of that time Toshiaki Murofushi, from whom I learned so much. My thoughts go also to the late Jean-Yves Jaffray and Ivan Kramosil, who were outstanding scientists in this domain and good friends.

I would like to thank many colleagues who have accepted to spend time in reading parts of this book. Needless to say, they greatly contributed to the quality of the book. In particular, many thanks are due to Alain Chateauneuf, Miguel Couceiro, Yves Crama, Denis Feyel, Peter Klement, Ehud Lehrer, Jean-Luc Marichal, Massimo Marinacci, Michel Maurin, Radko Mesiar, Pedro Miranda, Bernard Monjardet, Hans Peters, and Peter Sudhölter. Special thanks go to Ulrich Faigle for providing a proof of Theorem 3.24 and material on Walsh basis; Tomáš Kroupa for providing material on the Fourier transform and drawing my attention to the cone of supermodular games; Peter Wakker for invaluable comments on Chap. 5 (as well as on English!); Denis Bouyssou, Christophe Labreuche, Patrice Perny, and Marc Pirlot for in-depth discussion on Chap. 6; and finally to Thierry Denœux and Didier Dubois for fruitful discussion on Chap. 7 and drawing my attention to the possibilistic core, as well as to the ontic vs. epistemic view of sets.

This long task of writing would not have been possible without enough free time to do it and without the support of my institution. My sincere gratitude goes to Bernard Cornet, head of the research unit, and to Institut Universitaire de France, for having protected me against too many administrative and teaching tasks. Last but not least, countless thanks are due to my wife, Agnieszka Rusinowska, researcher in mathematical economics, for her unfailing support, understanding, and love, as well as for many comments on the last three chapters.

Despite all my efforts (and those of my colleagues), the book may contain typos, errors, gaps, and inaccuracies. Readers are encouraged to report them to me for future editions (if any), and all that remains for me now is to wish the readers a nice trip in the world of set functions.

Paris, France<br>January 2016

Michel Grabisch

# Contents


**1 Introduction** .......... 1
1.1 Notation .......... 1
1.2 General Technical Results .......... 3
1.3 Mathematical Prerequisites .......... 7
1.3.1 Binary Relations and Orders .......... 7
1.3.2 Partially Ordered Sets and Lattices .......... 8
1.3.3 Cones and Convex Sets .......... 13
1.3.4 Linear Inequalities and Polyhedra .......... 13
1.3.5 Linear Programming .......... 15
1.3.6 Cone Duality .......... 17
1.3.7 Support Functions of Convex Sets .......... 18
1.3.8 Convex Optimization and Quadratic Programming .......... 19
1.3.9 Totally Unimodular Matrices and Polyhedron Integrality .......... 20
1.3.10 Riesz Spaces .......... 21
1.3.11 Laplace and Fourier Transforms .......... 22

**2 Set Functions, Capacities and Games** .......... 25
2.1 Set Functions and Games .......... 26
2.2 Measures .......... 27
2.3 Capacities .......... 27
2.4 Interpretation and Usage .......... 28
2.4.1 In Decision and Game Theory .......... 28
2.4.2 In Operations Research .......... 30
2.5 Derivative of a Set Function .......... 32
2.6 Monotone Cover of a Game .......... 33
2.7 Properties .......... 34
2.8 Main Families of Capacities .......... 42
2.8.1 0-1-Capacities .......... 42
2.8.2 Unanimity Games .......... 42
2.8.3 Possibility and Necessity Measures .......... 43

2.8.4 Belief and Plausibility Measures ........ 44
2.8.5 Decomposable Measures ........ 44
2.8.6 $\lambda$-Measures ........ 46
2.9 Summary ........ 49
2.10 The Möbius Transform ........ 49
2.10.1 Properties ........ 52
2.10.2 Möbius Transform of Remarkable Games and Capacities ........ 54
2.11 Other Transforms ........ 58
2.12 Linear Invertible Transforms ........ 60
2.12.1 Definitions and Examples ........ 60
2.12.2 Generator Functions, Cardinality Functions ........ 62
2.12.3 Inverse of Cardinality Operators ........ 63
2.12.4 The Co-Möbius Operator ........ 64
2.12.5 The Interaction Operator ........ 65
2.12.6 The Banzhaf Interaction Operator ........ 69
2.12.7 Transforms of Conjugate Set Functions ........ 71
2.13 $k$-Additive Games ........ 73
2.14 $p$-Symmetric Games ........ 74
2.15 Structure of Various Sets of Games ........ 75
2.15.1 The Vector Space of Games ........ 75
2.15.2 The Cone of Capacities ........ 77
2.15.3 The Cone of Supermodular Games ........ 78
2.15.4 The Cone of Totally Monotone Nonnegative Games ........ 79
2.15.5 The Riesz Space of Games ........ 80
2.15.6 The Polytope of Normalized Capacities ........ 81
2.15.7 The Polytope of Belief Measures ........ 88
2.15.8 The Polytope of At Most k-Additive Normalized Capacities ........ 89
2.16 Polynomial Representations ........ 91
2.16.1 Bases of $\mathcal{PB}(n)$ ........ 92
2.16.2 The Fourier Transform ........ 96
2.16.3 Approximations of a Fixed Degree ........ 101
2.16.4 Extensions of Pseudo-Boolean Functions ........ 108
2.17 Transforms, Bases and the Inverse Problem ........ 117
2.17.1 Transforms and Bases ........ 117
2.17.2 The Inverse Problem ........ 123
2.18 Inclusion-Exclusion Coverings ........ 124
2.19 Games on Set Systems ........ 129
2.19.1 Case Where $X$ Is Arbitrary ........ 130
2.19.2 Case Where $X$ Is Finite ........ 134

**3 The Core and the Selectope of Games** ... 145
3.1 Definition and Interpretations of the Core ... 146
3.2 The Core of Games on $(N, 2^N)$ ... 148
3.2.1 Nonemptiness of the Core ... 148
3.2.2 Extreme Points of the Core ... 154
3.2.3 Additivity Properties ... 156
3.3 The Core of Games on Set Systems ... 157
3.3.1 Nonemptiness of the Core ... 157
3.3.2 Boundedness ... 158
3.3.3 Extremal Rays ... 161
3.3.4 Extreme Points ... 162
3.3.5 Faces ... 169
3.3.6 Bounded Faces ... 169
3.4 Exact Games, Totally Balanced Games, Large Cores and Stable Sets ... 174
3.5 The Selectope ... 181

**4 Integrals** ... 189
4.1 Simple Functions ... 190
4.2 The Choquet and Sugeno Integrals for Nonnegative Functions ... 191
4.3 The Case of Real-Valued Functions ... 196
4.3.1 The Choquet Integral ... 197
4.3.2 The Sugeno Integral ... 200
4.4 The Choquet and Sugeno Integrals for Simple Functions ... 202
4.4.1 The Choquet Integral of Nonnegative Functions ... 202
4.4.2 The Sugeno Integral of Nonnegative Functions ... 204
4.4.3 The Case of Real-Valued Functions ... 206
4.5 The Choquet and Sugeno Integrals on Finite Sets ... 207
4.5.1 The Case of Nonnegative Functions ... 207
4.5.2 The Case of Real-Valued Integrands ... 210
4.5.3 The Case of Additive Capacities ... 211
4.6 Properties ... 211
4.6.1 The Choquet Integral ... 212
4.6.2 The Sugeno Integral ... 227
4.7 Expression with Respect to the Möbius Transform and Other Transforms ... 234
4.7.1 The Choquet Integral ... 234
4.7.2 The Sugeno Integral ... 237
4.8 Characterizations ... 239
4.8.1 The Choquet Integral ... 239
4.8.2 The Sugeno Integral ... 244
4.9 Particular Cases ... 246
4.9.1 The Choquet Integral ... 246
4.9.2 The Sugeno Integral ... 251

4.10 The Choquet Integral on the Nonnegative Real Line ... 254
4.10.1 Computation of the Choquet Integral ... 254
4.10.2 Equimeasurable Rearrangement ... 258
4.11 Other Integrals ... 259
4.11.1 The Shilkret Integral ... 259
4.11.2 The Concave Integral ... 260
4.11.3 The Decomposition Integral ... 265
4.11.4 Pseudo-Additive Integrals, Universal Integrals ... 270
4.12 The Choquet Integral for Nonmeasurable Functions ... 272

**5 Decision Under Risk and Uncertainty ... 281**
5.1 The Framework ... 282
5.1.1 The Components of a Decision Problem ... 282
5.1.2 Introduction of Probabilities ... 284
5.1.3 Introduction of Utility Functions ... 285
5.2 Decision Under Risk ... 286
5.2.1 The Expected Utility Criterion ... 287
5.2.2 Stochastic Dominance ... 289
5.2.3 Risk Aversion ... 291
5.2.4 The Allais Paradox ... 292
5.2.5 Transforming Probabilities ... 293
5.2.6 Rank Dependent Utility ... 294
5.2.7 Prospect Theory ... 300
5.3 Decision Under Uncertainty ... 303
5.3.1 The Expected Value Criterion and the Dutch Book Argument ... 303
5.3.2 The Expected Utility Criterion ... 306
5.3.3 The Ellsberg Paradox ... 308
5.3.4 Choquet Expected Utility ... 309
5.3.5 Ambiguity and Multiple Priors ... 314
5.4 Qualitative Decision Making ... 317
5.4.1 Decision Under Risk ... 318
5.4.2 Decision Under Uncertainty ... 321

**6 Decision with Multiple Criteria ... 325**
6.1 The Framework ... 326
6.2 Measurement Theory ... 328
6.2.1 The Fundamental Problem of Measurement ... 328
6.2.2 Main Types of Scales ... 329
6.2.3 Ordinal Measurement ... 330
6.2.4 Difference Measurement ... 334
6.3 Affect, Bipolarity and Reference Levels ... 336
6.3.1 Bipolarity ... 337
6.3.2 Reference Levels ... 338
6.3.3 Bipolar and Unipolar Scales ... 339

6.4 Building Value Functions with the MACBETH Method ........... 341
6.4.1 The MACBETH Method .......... 341
6.4.2 Determination of the Value Functions .......... 342
6.5 Summary of the Construction of Value Functions .......... 344
6.6 The Weighted Arithmetic Mean as an Aggregation Function .......... 344
6.7 Towards a More General Model of Aggregation .......... 346
6.7.1 The Unipolar Case .......... 347
6.7.2 The Bipolar Case .......... 349
6.8 The Multilinear Model .......... 354
6.9 Summary on the Construction of the Aggregation Function .......... 357
6.10 Importance and Interaction Indices .......... 358
6.10.1 Importance and Interaction Indices for a Capacity .......... 358
6.10.2 Importance and Interaction Indices for an Aggregation Function .......... 360
6.10.3 A Statistical Approach: The Sobol' Indices .......... 363
6.10.4 The 2-Additive Model .......... 365
6.11 The Case of Ordinal Measurement .......... 367
6.11.1 The Emergence of the Sugeno Integral Model .......... 368
6.11.2 Monotonicity Properties of the Sugeno Integral Model .......... 370
6.11.3 Lexicographic Refinement .......... 372

**7 Dempster-Shafer and Possibility Theory** .......... 377
7.1 The Framework .......... 378
7.1.1 Dempster's Upper and Lower Probabilities .......... 378
7.2 Shafer's Evidence Theory .......... 379
7.2.1 The Case Where $m(\varnothing) > 0$ .......... 384
7.2.2 Kramosil's Probabilistic Approach .......... 385
7.2.3 Random Sets .......... 386
7.2.4 Ontic vs. Epistemic View of Sets .......... 391
7.3 Dempster's Rule of Combination .......... 391
7.3.1 The Rule of Combination in the Framework of Evidence Theory .......... 392
7.3.2 The Normalized and the Nonnormalized Rules .......... 394
7.3.3 Decomposition of Belief Functions into Simple Belief Functions .......... 397
7.4 Compatible Probability Measures .......... 398
7.5 Conditioning .......... 399
7.5.1 The General Conditioning Rule .......... 400
7.5.2 The Bayes' and Dempster-Shafer Conditioning Rules .......... 406
7.6 The Transferable Belief Model .......... 411
7.7 Possibility Theory .......... 413
7.7.1 The Framework .......... 414
7.7.2 Link with Dempster-Shafer Theory .......... 418

7.7.3 Links Between Possibility Measures and Probability Measures .... 419
7.7.4 The Possibilistic Core and Totally Monotone Anticore .... 423
7.8 Belief Functions and Possibility Measures on Lattices and Infinite Spaces .... 427
7.8.1 Finite Lattices .... 427
7.8.2 Infinite Spaces .... 437

**A Tables** .... 439
A.1 Bases and Transforms of Set Functions .... 439
A.2 Conversion Formulae Between Transforms .... 440

**List of Symbols** .... 443

**References** .... 451

**Index** .... 465