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Nonsmooth Mechanics

Models, Dynamics and Control

Third Edition

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To my son and my wife

Preface

Thank you for opening the third edition of this monograph. The first edition [202] was published in 1996 in the Lecture Notes in Control and Information Sciences series (vol. 220), and the second edition [203] in 1999 in the Communications and Control Engineering series, both at Springer Verlag London. The third edition, written almost 20 years after the first one, is a significantly revised and updated version. Indeed Nonsmooth Mechanics has witnessed intense research during the last two decades, in the fields of Applied Mathematics (existence and uniqueness of solutions, contact complementarity problem well-posedness, numerical analysis, bifurcation analysis), Mechanics (impact modeling, Painlevé paradoxes analysis), Systems and Control (regulation and trajectory tracking), Granular Matter, Robotics, etc. Software packages dedicated to nonsmooth mechanical systems also appeared here and there. It was therefore needed to report about all these novelties.

This book is devoted to the study of a class of nonsmooth dynamical systems of the general form:

$$\begin{cases} \dot{x}(t) \;=\; g(x(t), u) \\ f(x,t) \geq 0, \end{cases} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the system's state vector, $u \in \mathbb{R}^{n_u}$ is the vector of inputs, and the function $f(\cdot,\cdot)$ represents a set of m_u unilateral constraints which are imposed on the system. More precisely, the main topic is a subclass of such systems, namely mechanical systems subject to unilateral and bilateral constraints on the position (with or without friction), whose dynamical equations may be in a first instance written as:

$$\begin{cases} M(q(t))\ddot{q}(t) + F(q(t), \dot{q}(t), t, \lambda(t)) \;=\; 0 \\ f(q(t),t) \geq 0,\; \lambda_u(t) \geq 0,\; \lambda_u(t)^T f(q(t),t) \;=\; 0 \\ h(q(t),t) \;=\; 0, \end{cases} \tag{2}$$

where $q(t) \in \mathbb{R}^n$ is the vector of generalized coordinates of the system. The inertia matrix $M(q)$ will be assumed to be always symmetric, but not necessarily full rank (it may be positive semi-definite). The system may be constrained by a set of m_b bilateral constraints $h(q,t) = 0$ (this is the most common case in multibody dynamics). The vector function $f(q,t)$ represents a signed distance between the system and some environment, or more simply a condition for non-penetration between the bodies that constitute the system. The *contact forces* are represented through a *Lagrange multiplier* vector λ, which is split into λ_b for bilateral constraints, and λ_u for unilateral constraints. The multiplier λ_u satisfies a specific set of conditions with the distance function: they have to be both nonnegative (excluding penetrations between the bodies, as well as gluing effects, i.e., only nontensile contact interactions are modeled), and they have to be orthogonal one to each other (excluding distance effects like magnetic forces). These conditions are called *complementarity constraints*, and we will write them more compactly as:

$$0 \le f(q,t) \perp \lambda_u \ge 0, \tag{3}$$

where inequalities are understood componentwise, so that we may equivalently write $0 \le f_i(q,t) \perp \lambda_{u,i} \ge 0$ for each i. Complementarity is an ubiquitous concept all through this book. Mechanical systems composed of rigid bodies interacting with each other, fall into this subclass of systems, and may be named *nonsmooth multibody systems*. One particular feature of systems as in (2) is that they are of variable structure, or changing topology, because their dimension may vary due to complementarity constraints (from a certain point of view, this is similar to sliding mode controlled systems where attractive sliding surfaces make the system's dimension decrease or increase, and is met when Coulomb's friction or another tangent forces model imposes sticking modes).

Another feature of systems as in (1) and (2) is that their solutions are nonsmooth (with respect to time): nonsmoothness arises primarily from the occurrence of *impacts* (or *collisions*, or *percussions*) in the dynamical behavior, when the trajectories attain the surface $f(q,t) = 0$. They create velocity discontinuities, and are necessary to keep the trajectories within the subspace $\Phi(t) = \{q \in \mathbb{R}^n | f(q,t) \ge 0\}$ of the system's state space (or configuration space if one adopts a more geometrical point of view). Nonsmoothness may also be due to frictional effects, like when Coulomb's friction model is adopted: then the acceleration may suffer from discontinuities. It is therefore necessary, when dealing with such classes of dynamical systems, to focus on collision dynamics, with or without friction. But this is not sufficient: indeed, another important feature of systems as in (2) is their *hybridness*, where the word hybrid means that both continuous and discrete-event-like dynamics are mixed. Roughly speaking, the continuous dynamics are due to the vector field in (2), whereas the modes correspond to the algebraic constraints ($f(q,t)$ in (2) may be a vector) that may be active or inactive. Without going into further details at this stage (it is the goal of this monograph to provide a complete tour of such nonsmooth systems), let us already notice that the dynamics will generally be composed of ODEs, DAEs, MDEs[1] and finite automata. The particular feature of nonsmooth systems is that the automaton dynamics is ruled by the complementarity conditions. This renders their analysis so exciting, because it relies on complementarity theory, convex analysis, nonsmooth analysis, and variational inequalities. Actually, notice that nonsmooth models similar as the ones we shall describe here overstep the framework of mechanical systems, since they also apply for instance to electrical circuits [10].

What follows in this paragraph is a not an introduction to the history of nonsmooth phenomena study in mechanics. It only aims at briefly recalling some celebrated names who have been involved one way or another in this topic. The interested (French speaking) readers may have a look at [366, 1050, 1307] for a more complete exposition of history of mechanics. It is worth noting that the problems related to impact dynamics have attracted the interest of physicists for at least three centuries (much more if one includes the studies of ancient Greek engineers and mathematicians like Aristotle and Heron). In the "modern" times, a strong interest about shock phenomena was motivated by the well-known contest organized by the Royal Society of London in 1668. The impact physical laws were in particular discussed, studied, and used initially by scientists like[2] R. Descartes (F, 1596–1650), G. Leibniz (D, 1646–1716), I. Newton (UK, 1642–1727) [246, 925], Jacob Bernoulli [135] (CH, 1654–1705) [519], Jean le Rond d'Alembert (F, 1717–1783) [320] S.D. Poisson (F., 1781–1840) [1008], Ch. Huygens (NL, 1629–1695) [566], G. Coriolis (F., 1792–1843) [301, 302], J. Wallis (UK, 1616–1703), Ch. Wren (UK, 1632–1723), E. Mariotte (F, 1620–1684), L. Carnot (F, 1753–1823), H. Navier (F, 1785–1836) [920], MacLaurin (Scotland, 1698–1746) [920], the well-known Newton's and Poisson's restitution coefficients being still well alive as basic models for rigid bodies collisions. Shock processes were also widely used in the debates between Leibnizians and Newtonians or Cartesians [434, 571, 572], in their controversies about the definition of forces. The first book entirely dedicated to shock theory has been published by Edme Mariotte (F, 1620–1684) intitled *Traité de la Percussion ou Choc des Corps dans Lequel les Principales Règles du Mouvement, Contraires à celles que M. Descartes et quelques Autres Modernes ont Voulu Etablir, sont démontrées par leurs Véritables Causes* in 1673. He was inspired by Wallis, Huygens, and Wren.[3] Huygens wrote in *Projet Inachevé d'un Préface pour un Traité sur le Choc des Corps et la Force Centrifuge (1689)* that he was irritated by Mariotte and accused him of plagiarism:

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[1]Measure Differential Equations.

[2]In reality, it seems that the first "published" works on impact dynamics have been those of Thomas Hariot (around 1610–1620) [640] and the Dutch scientist Beeckman (around November–December 1618) who, contrarily to Descartes whose ideas on impact dynamics were almost all false, proposed theories that were not so incoherent when replaced in the early seventeenth century context [640, 1230].

[3]In *Mariotte, savant et philosophe (1684): analyse d'une renommée*, Librairie Philosophique J. Vrin, Paris, 1986.

"Mariotte a tout pris de moy… Je le luy dis un jour et il ne su que respondre." (Mariotte took everything from me… I told him once and he was not able to answer).

Later G. Darboux (F, 1842–1917) [326, 327], E.J. Routh (UK, 1831–1907) [1049], P. Appell (F, 1855–1930) [54], J.W. Gibbs (USA, 1839–1903) [446], A.M. Lyapunov (Ru, 1857–1918) [776], L. Poinsot (F, 1777–1859) [1006, 1007], and others [765, 904, 1265] worked on impact dynamics.[4] Although this fact has been a little forgotten now, rigid body (or more exactly particles) shock dynamics were extensively used in the seventeenth century to study light models [566] and also by artillerists [798] to predict the flight of cannon balls and their impacts. As we pointed out above, much of this scientific excitation was due to the will of the Royal Society of London whose scientists wanted to settle a coherent theory of motion.

Nonsmooth Mechanics belongs to Solid Mechanics. However, several other scientific communities have strong interests in this field. Applied Mathematicians, for problems related to existence and uniqueness of solutions, analysis of complex dynamics of certain impacting systems like billiards,[5] bifurcation analysis, researchers from Mechanical and Civil Engineering, as well as Physicists (the study of granular matter—sandpiles, gravels, planetary rings—has become a very important field that involves these three scientific communities), Robotics (to study the effect of impacts in the joints or the motion of the system after the impact, like in robot manipulators, bipeds, juggling or hopping robots, multifingered hands, $\cdots$), Electromechanics (electromechanical contacts are a major source of failures in many systems like automotives, aircraft, machine tools, consumer electronics, and therefore motivate the study of accurate models for simulation and design purposes), Computer Sciences (graphics, virtual reality) are scientific communities interested in nonsmooth multibody dynamical systems. These models are also used in Chemistry and Biology [285, 647, 1135, 1260, 1273], in Sports Dynamics for the analysis of tennis ball/racket or golf ball/club dynamics [55, 200, 201, 311, 597, 1113], and in Ecology for forest fire modeling [264, 339, 786].

I would like to end this introduction by mentioning two papers that have been, in my opinion, the most important ones in the field of "modern" nonsmooth mechanics:

G. Darboux, 1880 "Etude géométrique sur les percussions et le choc des corps," Bulletin des Sciences Mathématiques et Astronomiques, deuxième série, tome 4, pp. 126–160 and J.J. Moreau, 1988 "Unilateral contact and dry friction in finite freedom dynamics", in J.J. Moreau, P.D. Panagiotopoulos, (Eds.), *Nonsmooth Mechanics and Applications*, CISM Courses and Lectures no 302, International Centre for Mechanical Sciences, Springer-Verlag, pp. 1–82.

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[4]It is worth recalling that so many great scientists found an interest in impact dynamics. Indeed most of them are not known for their contributions in this field.

[5]In the literature, it seems that the word *vibro-impact systems* is used in the mechanical engineering field to name various types of systems that involve percussions. The word *billiards* refers to theoretical models of particles colliding in a closed domain, and is used mainly in mathematical physics.

The paper by the Mathematician Gaston Darboux (1842–1917) proposes a way to model the shock process and analytical developments that have been, and are still widely used in impact mechanics, more than one century later. The paper by Jean Jacques Moreau (1923–2014), who is one of the foundators of Convex Analysis[6] together with R.T. Rockfellar,[7] settles a general framework for the modeling of mechanical systems with unilateral constraints, based on convex analysis tools. It has motivated subsequent works on both the mathematical (well-posedness) and the numerical simulation sides (in particular concerning granular matter), which have considerable importance in this field.

This choice (both are French…) only reflects my own opinion. Finally, readers who want to learn more about frictionless multiple impact models should have a look at [929], and those who desire to learn about the numerical analysis and simulation of nonsmooth mechanical systems may read [13].

This book deals a lot with *modeling*. Let me quote the following:

Remember that all models are wrong; the practical question is how wrong do they have to be to not be useful. (in G.E.P. Box and N.R. Draper, Empirical Model-Building and Response Surfaces, 1987).

A deep (not superficial) understanding of engineering and physics is required to develop useful mathematical and computational models; the importance of models and their limitations is often given insufficient attention by control researchers. (N.H. McClamroch, IEEE Control Systems Magazine, October 2014).

The way a scientist may describe contact laws depends on his research area, and on the results he desires. [614]

Some authors, arguing that instantaneous forces do not exist, prefer not to use the notion of percussion and subsequent theory that determine their effects. There does not exist neither points nor straight lines in nature. Nevertheless we find such abstract objects useful and interesting. Certainly when passing to applications one has to quantify the errors that one may make by applying theorems derived from pure Science. But this problem is independent of the development of Science itself. [327]

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[6] *A Mechanician capable of doing Mathematics without any accent*, according to his own words.

[7] R.T. Rockafellar developed Convex Analysis with Mathematical Programming motivations, while J.J. Moreau did it with Nonsmooth Mechanics objectives in mind.

F. Génot, C. Glocker, D. Goeleven, M. Heemels, O. Huber, Y. Hurmuzlu, G. James, R. Kikuuwe, C. Lamarque, B.K. Le, R.I. Leine, C. Liu, M. Mata Jimenez, M. Monteiro-Marques, C.I. Morarescu, N.S. Nguyen, S.I. Niculescu, L. Paoli, C. Prieur, M. Schatzman, A. Tanwani, L. Thibault, A. Vieira, A. Zavala-Rio, H. Zhang, Z. Zhao.

Contents

Notation

- A matrix $M \in \mathbb{R}^{n\times n}$ is positive (semi) definite if $x^T Mx > 0$ for all $x \neq 0$ (resp. ≥ 0 for all x): $M \succ 0$ (resp. $M \succeq 0$). M is negative (semi) definite if $-M \succ 0$ (resp. $\succeq 0$): $M \prec 0$ (resp. $\preceq 0$). It is not necessarily symmetric.
- A matrix M is positive (resp. nonnegative) if all its entries M_{ij} are positive (resp. nonnegative).
- Let $f : \mathbb{R}^n \to \mathbb{R}^p$ be a differentiable function. Its Jacobian at x is $\frac{\partial f}{\partial x}(x) \in \mathbb{R}^{p\times n}$, its gradient $\nabla f(x) = \left(\frac{\partial f}{\partial x}(x)\right)^T \in \mathbb{R}^{n\times p}$.
- Let $x \in \mathbb{R}^n$, then $x > 0$ (resp. ≥ 0) means that each component $x_i > 0$ (resp. ≥ 0).
- The kinetic energy of a Lagrange system with generalized coordinate vector q, velocity $\dot{q}$, and inertia matrix $M(q) = M(q)^T \succeq 0$, is denoted $T(q,\dot{q}) = \frac{1}{2}\dot{q}^T M(q)\dot{q}$.
- The $n \times n$ identity matrix is I_n.
- Lexicographical inequalities: $(x_1, x_2, \cdots, x_n) \succcurlyeq 0$ means that if $x_1 = x_2 = \cdots = x_{i-1} = 0$ for $i-1 < n$, then $x_i \geq 0$ (the first nonzero entry is > 0, or all entries are zero); $(x_1, x_2, \cdots, x_n) \succ 0$ means that not all entries are zero, and the first nonzero entry $x_i > 0$.
- Let $x \in \mathbb{R}^n$, $M = M^T \succeq 0$ a $n \times n$ matrix, and $K \subseteq \mathbb{R}^n$ a nonempty set. The orthogonal projection of x on K in the metric defined by M is denoted as $\mathrm{proj}_M[K;x] \stackrel{\Delta}{=} argmin_{z\in K} \frac{1}{2} z^T Mz$. If K is convex and $M \succ 0$ it is unique.
- The boundary of a nonempty set Φ is denoted as $\mathrm{bd}(\Phi)$ (perhaps rarely as $\partial\Phi$ not to confuse with the subdifferential).
- The unit ball is $\mathbb{B} = \{x \in \mathbb{R}^n \mid ||x|| \leq 1\}$ where $||\cdot||$ is a norm.
- Let $u : \mathbb{R} \to \mathbb{R}^n$ be a function which has right and left limits at t. Then $\sigma_u(t) \stackrel{\Delta}{=} u(t^+) - u(t^-)$.
- Subscripts λ_{n} refer to normal direction, λ_{t} to tangential directions, while in λ_n the subscript $n \in \mathbb{N}$ is for a sequence of reals.

- The usual notation A: $\mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is employed for multivalued functions, i.e., functions such that $A(x)$ may be a subset of $\mathbb{R}^m$ for some $x \in \mathbb{R}^n$.
- A function such that $\left(\int_{[a,b]} |f(t)|^p dt\right)^{\frac{1}{p}} < +\infty$, $1 \leq p < +\infty$, is said to belong to the space $L^p([a,b])$ (or sometimes denoted $L_p([a,b])$. When $p = +\infty$, the space $L_\infty([a,b])$ contains functions such that ess sup$|f(t)| < +\infty$, i.e., functions which are bounded except on a set of measure zero.
- We say that $f \in C^p(I)$ if it is p times differentiable on I, and $f^{(p)}(\cdot)$ is continuous on I.
- diag$(x_1, x_2, \cdots, x_m)$, is the $m \times m$ diagonal matrix D with diagonal entry $D_{ii} = x_i$.
- Coefficients of restitution (CoRs): e_{n}: Newton (or kinematic) CoR, e_{p} Poisson (or kinetic) CoR, $e_{\star}$ energetic CoR.
- Given a matrix $A \in \mathbb{R}^{n \times n}$, its largest and smallest eigenvalues are denoted $\lambda_{\max}(A)$ and $\lambda_{\min}(A)$, respectively. Let $A \in \mathbb{R}^{n \times m}$, singular values are defined as $\sigma_i = \sqrt{\lambda_i(AA^T)} = \sqrt{\lambda_i(A^TA)}$ for $1 \leq i \leq \min(n,m)$, and the largest singular value is $\sigma_{\max}(A) = \sqrt{\lambda_{\max}(AA^T)}$. One has $\sigma_{\max}(A) = ||A||_2$ where $||\cdot||_2$ is an induced matrix norm, also denoted $||\cdot||_{2,2}$: $||A||_{2,2} = \max_{x \in \mathbb{R}^n \setminus \{0\}} \frac{||Ax||}{||x||}$, where $||\cdot||$ is the Euclidean norm.