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Antonio Cañada • Salvador Villegas

A Variational Approach to Lyapunov Type Inequalities

From ODEs to PDEs

Springer

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ISSN 2191-8198 ISSN 2191-8201 (electronic)
SpringerBriefs in Mathematics
ISBN 978-3-319-25287-2 ISBN 978-3-319-25289-6 (eBook)
DOI 10.1007/978-3-319-25289-6

Library of Congress Control Number: 2015953446

Mathematics Subject Classification (2010): 34B05, 34B15, 34C10, 34L15, 35J20, 35J25, 35J65, 49R05

Springer Cham Heidelberg New York Dordrecht London

Printed on acid-free paper

Springer International Publishing AG Switzerland is part of Springer Science+Business Media (www.springer.com)

To Ery and Patricia

Foreword

In his 1892 Doctoral thesis at Kharkov University on *The general problem of the stability of motion*, Aleksandr M. Lyapunov considered the stability of Hill's equation

$$x'' + a(t)x = 0, \tag{1}$$

where $a(t)$ is continuous and L-periodic. If we write $a \prec b$ when $a(t) \leq b(t)$ for all $t \in [0, L]$ and $a(t) < b(t)$ on a subset of positive measure, Lyapunov proved that if $0 \prec a$ and

$$\int_0^L a \leq \frac{4}{L}, \tag{2}$$

then all solutions of (1) are such that

$$x(L) = e^{\pm i\theta}x(0), \quad x'(L) = e^{\pm i\theta}x'(0) \tag{3}$$

for some $\theta > 0$ such that $\sin\theta \neq 0$ and, in particular, are bounded over $\mathbf{R}$. This is the first occurrence of what is called today *Lyapunov inequality*.

As variational equations for periodic solutions of nonlinear oscillators are of type (1), Lyapunov's result was important in the study of their stability. Many improvements, refinements, and generalizations of condition (2) followed almost immediately in Russia and elsewhere. An excellent description and bibliography can be found in Cesari's monograph *Asymptotic Behavior and Stability Problems in Ordinary Differential Equations*, Springer, 1963, and in Yakubovitch–Starzhinskii's two volumes of *Linear Differential Equations with Periodic Coefficients*, Halsted Press, 1975.

But inequality (2) is useful in other contexts than stability. Dealing in 1956 with the second-order nonlinear equation

$$x'' + R(t, x, x')x = Q(t, x, x') \tag{4}$$

with R, Q L-periodic with respect to t, continuous, and Q is bounded on $\mathbf{R}^3$, Volpato (*Rend. Sem. Mat. Univ. Padova* 25, 371–385) proved the existence of an L-periodic solution for (4) when $0 \le p(t) \le R(t,x,y) \le P(t)$, for all $(t,x,y) \in \mathbf{R}^3$ and some continuous L-periodic functions $0 \le p(t) \le P(t)$ such that $0 \prec p$ and

$$\int_0^L P \le \frac{4}{L}. \tag{5}$$

One recognizes in (5) Lyapunov inequality (2) for P. Those conditions imply that whenever a is continuous and L-periodic and such that $p(t) \le a(t) \le P(t)$ for all $t \in \mathbf{R}$, all solutions of (1) satisfy (3), and hence 0 is its only L-periodic solution. In 1964, Lasota and Opial (*Ann. Polon. Math.* 16, 69–94) replaced 4 by 16 in condition (5) and showed that 16 is the best possible constant. The existence of an L-periodic solution of (4) follows from a combination of those linear results with Schauder's fixed point theorem.

The existence of an L-periodic solution for (4) has been proved later when $p(t) \le R(t,x,y) \le P(t)$, for all $(t,x,y) \in \mathbf{R}^3$, with p and P such that $0 < \int_0^L p$, and

$$P \prec \frac{4\pi^2}{L^2} \tag{6}$$

or

$$\frac{4\pi^2 n^2}{L^2} \prec p \le P \prec \frac{4\pi^2(n+1)^2}{L^2} \tag{7}$$

for some integer $n \ge 1$.

Inequality (2) or (5) can be seen as L^1-norm conditions on a or P and inequality (6) or (7) as L^∞-norm conditions on p and P. One can therefore think about the possibility of Lyapunov-type inequalities in L^p-norm $\|\cdot\|_p$ $(1 \le p \le \infty)$ and of a possible unified treatment. One can also raise similar questions for other boundary conditions, like Dirichlet, Neumann, or anti-periodic ones, and investigate the possibility of extending the results to partial differential equations or systems of ordinary differential equations.

All those questions are elegantly and effectively answered in the monograph *A Variational Approach to Lyapunov-Type Inequalities* of Antonio Cañada and Salvador Villegas. Putting the main emphasis on Neumann boundary conditions

$$x'(0) = x'(L) = 0 \tag{8}$$

without neglecting other ones, they introduce and use an original and powerful variational approach to obtain the best L^p*-Lyapunov constant* for (1)–(8), namely,

$$\beta_p := \inf\{\|a^+\|_p : a \in L^p(0,L) \setminus \{0\}, \int_0^L a \geq 0,$$
$$(1)\text{–}(8) \text{ has nontrivial solutions}\}$$

Interestingly, β_p is achieved except for $p = 1$, provides the expected values $\beta_\infty = \frac{\pi^2}{L^2}$, $\beta_1 = \frac{4}{L}$, and an explicit formula is given for $p \in (1, \infty)$. Characterizations in terms of Rayleigh-type quotients are obtained as well. Those results, and similar ones for other boundary conditions, imply various existence and uniqueness results for corresponding nonlinear problems.

In order to generalize conditions of type (7) for the L^1-norm, the idea is nicely extended to higher order eigenvalues by introducing, for any integer $n \geq 1$, the L^1*-Lyapunov constants*

$$\beta_{1,n} = \inf\{a \in L^1(0,L) : \lambda_n = \frac{n^2\pi^2}{L^2} \prec a, (1)\text{–}(8) \text{ has nontrivial solutions}\}$$

If the determination of the β_p was done using essentially tools of the calculus of variations, that of the (non-achieved) $\beta_{1,n}$ is a striking combination of calculus of variations and hard analysis.

L^1 is rarely a good space for partial differential equations, and the fact is confirmed once more here. When defining and computing Lyapunov constant β_p for the Neumann problem

$$-\Delta u(x) = a(x)u(x) \ \text{ in } \ \Omega, \quad \frac{\partial u}{\partial n} = 0 \ \text{ on } \ \partial\Omega \tag{9}$$

with $\Omega \subset \mathbf{R}^N$ $(N \geq 2)$ a bounded regular domain and $a \in L^p(\Omega)$, the authors are faced with new situations, depending upon the relation between p and N. Indeed, $\beta_1 = 0$ for each $N \geq 2$, $\beta_p > 0$ for all $p \in (1, \infty]$ when $N = 2$ and, for $N \geq 3$, $\beta_p > 0$ if and only if $p \geq N/2$. Furthermore, β_p is achieved when $p > N/2$. A fine analysis of the dependence of β_p with respect to p is given, and the radial case provides refined results that are unreachable in the general case.

Everybody having experienced the use of variational methods knows that when they work for scalar differential equations, they work in general for some classes of differential systems. The last chapter of the book of Cañada and Villegas nicely illustrates this fact.

Mathematicians interested in the qualitative theory of linear differential equations will find in this monograph a number of interesting applications of Lyapunov inequalities and constants to the important question of disfocality. On the other hand, besides their mathematical interest, for example, in the theory of inverse problems, the linear equations with periodic coefficients have found important applications in quantum mechanics, since the pioneering work of Léon Brillouin.

Consequently, the book of Cañada and Villegas will be of interest for a substantial part of the mathematical community, from analysts to mathematical physicists. They will find there a modern, original, and elegant treatment of problems which, as hopefully shown by this Foreword, have their roots in classical papers on differential equations and are treated, in an elegant style, using the most recent techniques of functional analysis and the calculus of variations.

Louvain-la-Neuve, Belgium Jean Mawhin
April 2015

Preface

Different problems make the study of the so-called Lyapunov-type inequalities of great interest, both in pure and applied mathematics. Although the original historical motivation was the study of the stability properties of Hill's equation, other questions that arise in systems at resonance, crystallography, isoperimetric problems, Rayleigh-type quotients, oscillation and intervals of disconjugacy, etc. lead to the study of this type of inequalities for differential equations. This classical area of mathematics plays a significant role in the current research and remains a source of inspiration to this day.

In this book we examine in a detailed way some of the main aspects of this topic, including the most relevant results obtained by the authors in the last 12 years, as well as many other related results. Obviously, the selection of material is partly conditioned by the interest of the authors.

In our opinion, the contents of the book concerning higher eigenvalues, partial differential equations, and systems of equations are particularly innovative and through the whole monograph, an especial emphasis is done in the variational characterization of the best Lyapunov constants. This unified variational point of view makes possible the study of many cases, featuring a systematic discussion of different types of equations and boundary conditions, both for ordinary and partial differential equations. The applications include nonlinear resonant problems, the study of the stability of linear periodic equations (both for scalar and systems of equations), and the analysis of the sign of the eigenvalues of certain eigenvalue problems.

This work can be considered self-contained, with detailed proofs and a special emphasis on motivation and understanding of the basic ideas. Taking in mind a balanced presentation of both pure and applied aspects, we have tried to write this work in a style accessible to a broad audience, although a great variety of methods from classical analysis, differential equations, and nonlinear functional analysis are used. However, some proofs (especially those referring to the PDE case) are particularly laborious.

The book is addressed to experienced researchers working in the subject and to young researchers who want to start on these topics. The expository content, with detailed proofs and an appropriate list of references in each chapter, brings the reader quickly to the forefront of research. The volume contains numerous explanatory notes on the showed results and their relation to the existing literature.

Granada, Spain
April 2015

Antonio Cañada
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Acknowledgments

Our deepest gratitude to our friend and colleague Jean Mawhin for writing the foreword. His influence on the research group on differential equations at the Department of Mathematical Analysis of the University of Granada has been, along 35 years, decisive in all the aspects. We are also very grateful to our colleagues: G. López, J.A. Montero, and P. Torres who read this text and made useful comments and suggestions. We want to thank the referees for their constructive comments. The editorial and production staff of Springer have always been ready to offer a very efficient assistance. Finally, this work has been supported by the project MTM2012.37960 of the Spanish Science and Innovation Ministry.

Contents

Acronyms and Notation

$\mathbf{N}$	The set of natural numbers
$\mathbf{R}$	The set of real numbers
b.v.p.	Boundary value problem
ODEs	Ordinary differential equations
PDEs	Partial differential equations
$L^p(0,L)$ $1 \le p < \infty$	The Lebesgue space of measurable functions $a(\cdot)$ such that $\lvert a(\cdot)\rvert^p$ is integrable in $(0,L)$
$L^\infty(0,L)$	The Lebesgue space of measurable functions such that there exists a constant c satisfying $\lvert a(x)\rvert \le c$, almost everywhere (a.e.) in $(0,L)$
$L^p(\Omega)$ $1 \le p < \infty$	The Lebesgue space of measurable functions $a(\cdot)$ such that $\lvert a(\cdot)\rvert^p$ is integrable in Ω, a bounded and regular domain in $\mathbf{R}^N$
$L^\infty(\Omega)$	The Lebesgue space of measurable functions such that there exists a constant c satisfying $\lvert a(x)\rvert \le c$, almost everywhere (a.e.) in Ω
$L_T(\mathbf{R},\mathbf{R})$	The set of T-periodic functions $a(\cdot)$ such that $a\vert_{(0,T)} \in L^1(0,T)$
$\lVert \cdot \rVert_p$	The usual norm in the spaces L^p
$\langle \cdot,\cdot \rangle$	The usual scalar product in $\mathbf{R}^n$
$H^1(0,L)$	The usual Sobolev space on the interval $(0,L)$
$H^1(\Omega)$, $H^1_0(\Omega)$	The usual Sobolev spaces on a bounded and regular domain $\Omega \subset \mathbf{R}^N$
$\dfrac{\partial}{\partial n}$	Outer normal derivative on $\partial\Omega$
$W^{m,p}(\Omega)$	The usual Sobolev space on a bounded and regular domain $\Omega \subset \mathbf{R}^N$
$H^1_T(0,T)$	The subset of T-periodic functions of the Sobolev space $H^1(0,T)$
B_r	The open ball in $\mathbf{R}^N$ of center zero and radius r

$C_T(\mathbf{R},\mathbf{R})$	The set of real T-periodic and continuous functions defined in $\mathbf{R}$
$\mathscr{M}(\mathbf{R})$	The set of real $n \times n$ matrices
β_p	Best L_p Lyapunov constant
f_u	Partial derivative of the function f with respect to u
$c \prec d$	$c, d \in L^1(0,L),\ c(x) \le d(x)$ for a.e. $x \in [0,L]$ and $c(x) < d(x)$ on a set of positive measure
$C \le D$	The relation $C \le D$ between $n \times n$ matrices means that $D - C$ is positive semi-definite
a^+	The positive part of the function a, i.e., $a^+(t) = \max\{0, a(t)\}$
$\det A$	The determinant of the matrix A
$\rho(A)$	The spectral radius of the matrix A
Trace (A)	The trace of the matrix A
Δ	The Laplacian operator