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The Analysis and Geometry of Hardy's Inequality

Springer

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To
Helen, Mari and Barbara

Preface

This book is a study of the many refinements and incarnations of the Hardy inequality

$$\int_{\Omega} |\nabla u(\mathbf{x})|^p d\mathbf{x} \geq C(p,\Omega) \int_{\Omega} \frac{|u(\mathbf{x})|^p}{\delta(\mathbf{x})^p} d\mathbf{x}, \quad u \in C_0^{\infty}(\Omega), \tag{1}$$

where Ω is a domain (an open connected set) in $\mathbb{R}^n$, $n \geq 1$, with non-empty boundary $\partial\Omega$, $\delta(\mathbf{x})$ is the distance from $\mathbf{x} \in \Omega$ to $\partial\Omega$, $1 \leq p < \infty$, and $C(p,\Omega)$ is a positive constant depending on p and Ω in general. The original continuous form of the inequality was for $\Omega = (0,\infty)$, $\delta(x) = |x|$, and appeared in [72], having been motivated by work of Hardy on a discrete analogue and a double series inequality of Hilbert. It attracted the attention of other mathematicians, notably Landau, and was highlighted by Hardy et al. in [75]; see [91] for a detailed account of the history. In its many guises, the inequality has played an important role in mathematical analysis and mathematical physics, which is way beyond what could have been expected at its humble beginning. Extensions and refinements to a multitude of function spaces have been studied extensively, which, apart from their intrinsic interest, have had significant implications for the function spaces and the relationships between them, and important applications to differential equations. The case $p = 2$, $\Omega = \mathbb{R}^n \setminus \{0\}, \delta(\mathbf{x}) = |\mathbf{x}|$ of (1) is a mathematical representation of Heisenberg's uncertainty principle in quantum mechanics, which asserts that the momentum and position of a particle can't be simultaneously determined. Furthermore, the spectral analysis of quantum mechanical systems involving Coulomb forces between constituent particles features this $L^2(\mathbb{R}^n)$ version of Hardy's inequality in a natural way. In his book *A Mathematician's Apology* [74], Hardy expresses the view that for a theorem to be significant, it must have both *generality* and *depth*. He also asserts that nothing he had ever done was *useful*. The role of his inequality in mathematics undoubtedly confirms his notion of "significance", while its implications in quantum mechanics, with its tentacles affecting every aspect of modern life, would contradict Hardy's feeling about its "uselessness".

In the first two sections of Chap. 1, a general form of the Hardy inequality is proved, initially in $\mathbb{R}_+ := (0, \infty)$ and subintervals (a, b) of $\mathbb{R}$ and then in $\mathbb{R}^n$ for $n \geq 2$, optimal constants being obtained. The rest of Chap. 1 is a cornucopia of techniques and results, which will be of subsequent importance. These include a brief description of Sobolev spaces and the inequalities of Sobolev, Friedrichs and Poincaré; Fourier transforms, rearrangements and their application to making a comparison of the Hardy and Sobolev inequalities; the Cwikel, Lieb, Rosenbljum (CLR) inequality concerning the number of negative eigenvalues of the Dirichlet Laplace operator in $L^2(\Omega)$ and a comparison of the CLR and the appropriate Sobolev inequality; Kato's inequality and relativistic analogues of Hardy's inequality.

Chapter 2 is on properties of a general domain Ω, which are of significance to the function δ. For instance, the **skeleton** $\mathcal{S}(\Omega)$ is the subset of Ω consisting of points, which are equidistant from more than one point on the boundary, and this coincides with the set of points at which δ is not differentiable. Another important subset of Ω is the **ridge**, or **central set**, $\mathcal{R}(\Omega)$, which lies between $\mathcal{S}(\Omega)$ and its closure. It is shown that the closure of the ridge is the **cut locus**, which is a concept used extensively by Li and Nirenberg in [108]. Properties of δ when Ω is convex, or $\mathbb{R}^n \setminus \overline{\Omega}$ is convex, are established, and when Ω is a domain with smooth boundary, an explicit formula for $\Delta\delta(\mathbf{x})$ is determined for all $\mathbf{x} \in \Omega \setminus \overline{\mathcal{R}(\Omega)}$. The principal curvatures of a C^2 boundary, and the mean curvature of $\partial\Omega$, feature prominently in the second part of the chapter, and indeed, the rest of the book.

The study of inequalities of type

$$\int_\Omega |\nabla u(\mathbf{x})|^p d\mathbf{x} \geq C(p, \Omega) \int_\Omega \left(\frac{|u(\mathbf{x})|^p}{\delta(\mathbf{x})^p} + a(\mathbf{x}, \delta(\mathbf{x}))|u(\mathbf{x})|^p \right) d\mathbf{x} \tag{2}$$

starts in earnest in Chap. 3, the case $a = 0$ being of particular interest and referred to as Hardy's inequality on Ω. We begin with a list of some important results in the literature to set the scene, which bring in, *inter alia*, the notions of capacity and fatness, the Hausdorff and Aikawa dimensions of the boundary, and the mean distance function $\delta_{M,p}$ introduced by Davies in [41] in the case $p = 2$ for an arbitrary Ω. Included subsequently are a proof of the optimal constant for a convex domain Ω, and of Ancona's lower bound for the constant $C(2, \Omega)$ when Ω is a simply connected planar domain. Some of the main results in this chapter are based on ones from [20, 107], using tools developed in the previous chapter. Of special note is the result from [107] that if Ω has a C^2 boundary and a non-positive mean curvature (a so-called *weakly mean convex domain*), then

$$\mu_p(\Omega) := \inf_{C_0^\infty(\Omega)} \frac{\int_\Omega |\nabla f|^p d\mathbf{x}}{\int_\Omega |f/\delta|^p d\mathbf{x}} = \left(\frac{p-1}{p}\right)^p, \tag{3}$$

which extends a well-known result for convex domains.

In [118], Maz'ya proved the inequality

$$\int_{\mathbb{R}^n_+} \left(|\nabla u|^2 - \frac{|u|^2}{4x_n^2} \right) d\mathbf{x} \geq K_{n,2} \left(\int_{\mathbb{R}^n_+} |u|^{\frac{2n}{n-2}} d\mathbf{x} \right)^{\frac{n-2}{n}}, \tag{4}$$

for $u \in C_0^\infty(\mathbb{R}^n_+)$, where $\mathbb{R}^n_+$ is the half-space $\mathbb{R}^{n-1} \times \mathbb{R}_+$ and $\mathbf{x} = (\mathbf{x}', x_n)$, $\mathbf{x}' \in \mathbb{R}^{n-1}$, $x_n \in \mathbb{R}_+$. This is the prototype of the Hardy-Sobolev-Maz'ya (HSM) inequalities, which combine elements of both the Hardy and Sobolev inequalities, and Chap. 4 is devoted to them. We present the following result of Frank and Loss from [62] for a general domain $\Omega \subsetneq \mathbb{R}^n$, in which the mean distance function $\delta_{M,p}$ plays the role of the distance function δ : there exists a constant $K_{n,p}$, depending only on n and p, such that for all $u \in C_0^\infty(\Omega)$ and $p \geq 2$,

$$\int_\Omega \left(|\nabla u|^p - \left(\frac{p-1}{p} \right)^p \frac{|u|^p}{\delta_{M,p}^p} \right) d\mathbf{x} \geq K_{n,p} \left(\int_\Omega |u|^{\frac{np}{n-p}} d\mathbf{x} \right)^{\frac{n-p}{n}}. \tag{5}$$

If Ω is convex, $\delta_{M,p} \leq \delta$, and (5) becomes an extension of the HSM inequality obtained by Filippas et al. in [60] for a bounded convex domain Ω with a C^2 boundary and $p = 2$, and answers in the affirmative their query if the constant can be chosen to be independent of Ω. Chapter 4 also includes HSM inequalities featuring the mean curvature of the boundary of Ω, and one of Gkikas in [69] for exterior domains.

The first part of Chap. 5 is on Schrödinger operators involving magnetic fields of Aharonov-Bohm type. The Laptev-Weidl inequality in $L^2(\mathbb{R}^2)$ is derived, followed by related Sobolev and CLR inequalities. Hardy-type inequalities for Aharonov-Bohm magnetic fields with multiple singularities are proved, and also a generalised Hardy inequality for magnetic Dirichlet forms. Finally in Chap. 5, there is a discussion of Pauli operators in $\mathbb{R}^3$ with magnetic fields, and inequalities of Hardy, Sobolev and CLR type are proved to exist if the Pauli operator has no zero modes.

Chapter 6 is concerned with the Rellich inequality

$$\int_{\mathbb{R}^n} |\Delta u(\mathbf{x})|^2 d\mathbf{x} \geq \frac{n^2(n-4)^2}{16} \int_{\mathbb{R}^n} \frac{|u(\mathbf{x})|^2}{|\mathbf{x}|^4} d\mathbf{x}. \tag{6}$$

A proof of an $L^p(\mathbb{R}^n)$ version of the inequality is given, based on that of Davies and Hinz in [45], and this is followed by a Rellich-Sobolev inequality in $L^2(\Omega)$ for a domain $\Omega \subset \mathbb{R}^n$ due to Frank (private communication, 2007). Inequalities involving Aharonov-Bohm type magnetic potentials in $L^2(\mathbb{R}^n)$ are established, which are analogous to the Laptev-Weidl inequality of Chap. 5, and a CLR-type inequality for associated bi-harmonic operators is proved.

The book is primarily designed for the mathematician, but we hope that it will also appeal to the scientist who has an interest in quantum mechanics. A good basic knowledge of real and complex analysis is a prerequisite. Also, familiarity with the Lebesgue integral, spectral analysis of differential operators, and elementary differential geometry would be helpful, but only the barest essentials of these areas are assumed, and background information is always provided; where necessary, precise references to the literature are given.

Chapters are divided into sections and sections are sometimes divided into subsections. Theorems, Corollaries, Lemmas, Propositions, Remarks and equations are numbered consecutively. At the end of the book, there are author, subject and notation indices.

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Basic Notation

$\mathbb{R}$:	Real numbers
$\mathbb{R}^n$:	n-Dimensional Euclidean space
$\mathbb{N}$:	Positive integers
$\mathbb{N}_0 = \mathbb{N} \cup \{0\}$	
$\mathbb{Z}$:	Integers
$\mathbb{C}$:	Complex numbers
Ω :	Domain—a connected open subset of $\mathbb{R}^n$
$\partial\Omega$:	Boundary of Ω
$\overline{\Omega}$:	Closure of Ω
$\rightharpoonup$:	Weak convergence
$X \hookrightarrow Y$:	X is continuously embedded in Y
$L^p(\Omega),\ 1 \leq p < \infty$:	Lebesgue space of functions f with $\lvert f\rvert^p$ integrable on Ω
$\\| \cdot \\|_p$ or $\\| \cdot \\|_{p,\Omega}$:	Norm on $L^p(\Omega)$
$l^p,\ 1 \leq p < \infty$:	Space of sequences $\{x_n\}_{n\in\mathbb{N}}$ such that $\sum_{n=1}^{\infty} \lvert x_n\rvert^p < \infty$
$W^{k,p}(\Omega), H^{k,p}(\Omega)$:	Sobolev spaces
$C_0^\infty(\Omega)$:	Infinitely differentiable functions with compact supports in Ω
$W_0^{k,p}(\Omega)$:	Closure of $C_0^\infty(\Omega)$ in $W^{k,p}(\Omega)$
$\omega_n = \pi^{n/2}/\,\Gamma(1 + n/2)$:	Volume of unit ball in $\mathbb{R}^n$

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