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George A.F. Seber

# The Linear Model and Hypothesis

A General Unifying Theory

 Springer

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# Preface

In 1966 my monograph *The Linear Hypothesis: A General Theory* was published as one in a series of statistical monographs by Griffin, London. Part of the book arose out of my PhD thesis, which took a more general approach than usual to linear models. It used the geometrical notion of projections onto vector spaces using idempotent matrices, thus providing an elegant theory that avoided being involved with ranks of matrices. Although not a popular approach at the time, it has since become an integral part of theoretical regression books where least squares estimates, for example, are routinely given a geometrical interpretation.

Over the years I have written extensively on related topics such as linear and nonlinear regression, multivariate analysis, and large sample tests of general hypotheses including, for example, those arising from the multinomial distribution. Given this additional experience and the fact that my original monograph is now out of print, the time has come to rewrite it. This is it! Initially the 1966 monograph was written as an attempt to show how the linear model and hypothesis provide a unifying theme where all hypotheses are either linear or asymptotically so. This means that the linear theory can be applied in a variety of modeling situations and this monograph extends the breadth of these situations. In a monograph of this size, the emphasis is on theoretical concepts, and the reader needs to look elsewhere for practical applications and appropriate software. I appreciate that these days the focus of statistical courses is much more applied. Numerous computationally oriented books have been written, for example, on using the statistical package R that was originally developed in the Statistics Department here at University of Auckland. However I would mention that my books on linear, nonlinear, and multivariate models all have comprehensive chapters on computational details and algorithms, as well as practical examples.

Who is the monograph for? It is pitched at a graduate level in statistics and assumes that the reader is familiar with the basics of regression analysis, analysis of variance, and some experimental designs like the randomized block design, with brief extensions to multivariate linear models. Some previous exposure to nonlinear models and multinomial goodness-of-fit tests will help, and some knowledge of the multivariate normal distribution is assumed. A basic knowledge of the matrix theory

is assumed throughout, though proofs of most of the matrix results used are given either in the text or in the Appendix. My aim is to provide the reader with a more global view of modeling and show connections between several major statistical topics.

Chapters 1, 2, 3 and 4 deal with the basic ideas behind the book: Chap. 1 gives some preliminary mathematical results needed in the book; Chap. 2 defines the linear model and hypothesis with examples; Chap. 3 is on estimation; and Chap. 4 is on hypothesis testing, all from a geometrical point of view. Chapter 5 looks at some general properties of the  $F$ -test, and in Chap. 6 methods of testing several hypotheses are discussed. Chapters 7, 8 and 9 look at special topics: Chap. 7 is about augmenting hypotheses as in analysis of covariance and missing observations, for example, Chap. 8 looks at nonlinear models and Chap. 9 at multivariate models. Chapters 10, 11 and 12 involve considerable asymptotic theory showing how general hypotheses about sampling from general distributions are asymptotically equivalent to corresponding linear theory. The book finishes with an appendix giving some useful, and in some cases not so common, matrix results with proofs.

Looking back after having been retired for a number of years, I am grateful for the stimulus given to my writing through teaching most of the topics mentioned above at University of Auckland, New Zealand. Teaching certainly clarifies one's understanding of a subject. In conclusion I would like to thank two referees for their helpful comments.

Auckland, New Zealand  
February 2015

George A.F. Seber

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