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Ihsen Yengui

Constructive Commutative Algebra

Projective Modules Over Polynomial Rings
and Dynamical Gröbner Bases



Springer

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