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László Fuchs

Abelian Groups

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László Fuchs
Mathematics Department
Tulane University
New Orleans, LA, USA

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Dedicated with love
to my wife Shula,
our daughter Terry, and son David

Preface

The theory of abelian groups is a branch of (abstract) algebra, which deals with commutative groups, named after the Norwegian mathematician Niels H. Abel. Curiously enough, it is rather independent of general group theory: its methods bear only a slight resemblance to the non-commutative case. (However, there is a close relationship to the theory of modules, especially over integral domains.)

This is my third book on abelian groups. My first book was published in 1958; it was a reasonably comprehensive systematic summary of the theory at that time. With the advent of homological methods and the explosion of the theory of torsion groups, an extended version was needed that was published in 1970 and 1973. Even two volumes could not claim any more a comprehensive status, but they seem to have presented accurately the main streams in the theory. Today, after four decades of developments and thousands of publications, it is hardly imaginable to have a comparably complete volume. As a consequence, I had to scale down my goals. By no means could I be content with just a broad introductory volume, so I had in mind a monograph which goes much beyond being a mere introduction and concentrates on most of the advanced ideas and methods of today's research. I do hope that this volume will provide a thorough and accurate picture of the current main trends in abelian group theory.

The audience I have in mind with this book is anybody with a reasonable mathematical maturity interested in algebra. My aim has been to make the material accessible to a mathematician who would like to study abelian groups or who is looking for particular results on abelian groups needed in his/her field of specialization. I have made an effort to keep the exposition as self-contained as possible, but I had to be satisfied with stating a few very important theorems without proofs whenever the proofs would not fit in this volume. The specific prerequisites are sound knowledge of the rudiments of abstract algebra, in particular, basic group and ring theory. Also required is some exposure to category theory, topology, and set theory. A few results where more advanced set theory is indispensable are included for those who are willing to indulge in a bit more sophisticated set theory. In order to fill the need of a reference source for experts, much additional information about the topics discussed is included also in the "Notes" at the end of sections with reference to relevant publications. Many results discussed are collected from scattered articles in the literature; some theorems appear here for the first time in a book.

The writing of such a book requires, inevitably, making tough choices on what to include and what to leave out. I have tried to be selective so that the central ideas stand out. My guiding principle has been to give preference to important methods and typical results as well as to topics, which contain innovative ideas or which I felt were particularly instructive. It is no secret that several flourishing areas of module theory have their origin in abelian groups where the situation is often more transparent—I have paid special attention to these results as well. As I have tried to make the book more accessible for starters, generality is sometimes sacrificed in favor of an easier proof. I have shied away from theorems with too technical proofs unless the result, I believe, is theoretically or methodically extremely important. In the selection of additional topics, the guide was my personal interest (which I view as an indisputable privilege of authors).

In order to give a fair idea of the current main streams in abelian group theory, it is impossible to ignore the numerous undecidable problems. Abelian group theory is not only distinguished by its rich collection of satisfactory classification theorems, but also for having an ample supply of interesting undecidable problems. Shelah's epoch-making proof of the undecidability of Whitehead's problem in ZFC marked the beginning of a new era in the theory of abelian groups with set-theoretical methods assuming the leading role. The infusion of ideas from set theory created as radical a change in the subject as Homological Algebra did a quarter of a century before. Therefore, I had to take a more penetrating approach to set-theoretical methods and to discuss undecidable problems as well, but I treat these fascinating problems as interesting special cases of theoretical importance, rather than as a main object of a systematic study. Though a short survey of set-theoretical background is given, the reader is well advised to consult other sources.

I am a bit leisurely, primarily in the first half of the book, with the method of presentation and the proofs in order to assist students who want to learn the subject thoroughly. I have included a series of exercises at the end of each section. As is customary, some of them are simple to test the reader's comprehension, but the majority give noteworthy extensions of the theory or sidelines to enrich the topics studied in the same section. A student should conscientiously solve several exercises to check how he/she commands the material. Serious attempts have been made to provide ample examples to illustrate the theory. Good examples not only serve to motivate the results, but also provide an explicit source of ideas for further research.

Past experience suggests that listing open problems is a good way of encouraging young researchers to start thinking seriously on the subject. In view of this, I am listing open problems at the end of each chapter, with a brief commentary if needed. I have not given any serious thought to their solutions; so a lucky reader may find quick answers to some of them.

This volume follows basically the same line of development as my previous books on abelian groups. With the kind permission of the publishing company Elsevier, I could include portions of my two volume book "Infinite Abelian Groups." The book was out of print by 1990, and at that time I was playing with the idea of revising it, but decided to join forces with my friend Luigi Salce to write a monograph on modules over non-noetherian integral domains. When I felt ready to get involved in working on a book on abelian groups, Hurricane Katrina interfered: it forced us to leave our home for three-and-a-half years. It took me some time to brace myself for such a big task as writing a book. When I realized that a mere revision would not be satisfactory, I had no choice but to start working on a new book.

The first two chapters of this volume are introductory, and the reader who is knowledgeable in group theory will want to skip most of the sections very quickly. For those who are not so familiar with more advanced set-theoretical methods, it might help to read Sect. 4 carefully before proceeding further. The text starts in earnest in Chapter 3, which provides a thorough discussion of direct sums of cyclic groups. Chapter 4 is devoted to divisibility and injectivity, while Chapter 5 explores pure subgroups along with basic subgroups. In Chapter 6, we introduce algebraically compact groups motivated by purity and topological compactness.

In the next three chapters, we embark on the fundamental homological machinery needed for abelian groups. The insight developed here is essential for the rest of the book. We conclude Chapter 9 with the theory of cotorsion groups.

Armed with the powerful tools of homological algebra, we proceed to the next six chapters, which form the backbone of the theory, exploring the structure of various classes of abelian groups. The structure theory is a vast field of central importance that is not easy to organize into a coherent scheme. Starting with torsion groups, we first deal with p-groups that contain no elements of infinite height. Then we go on to explore the Ulm-Zippin theory of countable p-groups leading to the highlights of the impressive theory of totally projective p-groups. Chapters 12 and 13 provide a large amount of material on torsion-free groups—an area that has shown great advancement in the past quarter of century. We explore indecomposable groups, slender groups, and vector groups. The discussion culminates in the proof of the undecidability of the famous Whitehead problem. Chapter 14 serves as a broad introduction to the fascinating theory of Butler groups. In Chapter 15, the main results on mixed groups are presented.

The final three chapters deal with endomorphism rings, automorphism groups, and the additive groups of rings. Some ideas are introduced that interact between abelian groups and rings.

I stress that the reader should by no means take this book as a complete survey of the present state of affairs in abelian group theory. A number of significant and more advanced results in several areas of the theory, as well as a wealth of important topics (like group algebras) are left out, not to mention topics like the more advanced theory of finite and infinite rank torsion-free groups, as well as p-groups more general than totally projective groups. In spite of all these, I sincerely hope that the material discussed provides a significant amount of information that will open up new and promising vistas in our subject.

As far as the bibliography is concerned, references are not provided beyond the list of works quoted in the text and in the "Notes." No reference is given to the exercises. This self-imposed restriction was necessary in view of the vast literature on abelian groups. The reader who is interested in going beyond the contents of this book should use the list of references as a guide to other sources. References outside the theory of abelian groups are usually given in an abbreviated form embedded in the "Notes." The system of cross references is simple: lemmas, theorems etc. are numbered in each chapter separately. Books are cited by letters in square brackets. I have attempted to give credit to the results, and I apologize if I have made mistakes or omissions.

A remark is in order about notational conventions. We are using the functional notation for maps, thus $\phi(x)$, or simply ϕx, is the image of x under the map ϕ. Accordingly, the product $\phi\psi$ of two maps is defined as $(\phi\psi)(x) = \phi(\psi(x))$. This is not a universally adopted convention in abelian group theory, but is predominant. The "Table of Notations" should be consulted if symbols are not clear. I have introduced new notation or terminology only in a few places where I found the frequently used terms clumsy or confusing, or if a name was missing.

I am grateful to my friends and colleagues Lutz Strüngmann, Luigi Salce, Kulumani M. Rangaswamy, Claudia Metelli, Brendan Goldsmith, and Ulrich Albrecht for their comments on portions of an earlier version of the manuscript. I have greatly benefitted from the comments I received from them. I apologize for the errors which remain in the text.

Metairie, Louisiana, USA László Fuchs
March 27, 2015

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Table of Notations

Set Theory

$\subseteq, \subset$: subset, proper subset
$\cup, \cap$: set union, intersection
$\times$: cartesian product
$X \setminus Y = \{x \in X \mid x \notin Y\}$
$|S|$: cardinality of the set S
$\varnothing$: empty set
$\{a \in A \mid *\}$: set of elements of A satisfying $*$
$\{a_i\}_{i\in I}$: the set of all a_i with $i \in I$
$\kappa, \lambda, \mu, \dots$: cardinals
$\alpha, \beta, \dots, \rho, \sigma, \tau$: ordinals
$\aleph_{-1}$: finite, $\aleph_0$: countable cardinal
$2^{\aleph_0}$: continuum
$\aleph_\sigma$: σth infinite cardinal, σth aleph
ω: first infinite ordinal or the set $\{0, 1, \dots, n, \dots\}$
ω_σ: σth initial ordinal ($|\omega_\sigma| = \aleph_\sigma$)
cf σ: cofinality of the ordinal σ
$\Rightarrow$: implication
$\Leftrightarrow$: equivalence, if and only if
$\forall$: for all
$\exists$: there exists
$\neg$: negation
ZFC : Zermelo-Fraenkel axioms of set theory + Axiom of Choice
CH: Continuum Hypothesis
GCH: Generalized Continuum Hypothesis
V: model of set theory
L: Gödel's Axiom of Constructibility
V = L: Axiom L is assumed
$\diamondsuit$: Jensen's Diamond Principle

MA: Martin's Axiom
$\mathcal{A}b$: category of abelian groups
$\mathcal{C}, \mathcal{D}, \ldots$: categories
$\mathcal{V}, \mathcal{V}_p$: category of valuated groups or vector spaces
$\mathcal{G}, \mathcal{H}, \ldots$: $G(\kappa), H(\kappa)$-families of subgroups

Maps

$\to$: mapping, homomorphism
$A \xrightarrow{\alpha} B$: map α from A to B
$\mapsto$: corresponds to
$\dot{\to}$: quasi-homomorphism
$\mathbf{1}_A$: identity map on A
$\dot{n}$: multiplication by integer n
$\phi \restriction A$: restriction of map ϕ to A
Im ϕ: image of map ϕ
Ker ϕ: kernel of map ϕ
Coker ϕ: cokernel of map ϕ
Δ, ∇: diagonal, codiagonal map
$\oplus\, \phi_i$: direct sum of maps ϕ_i
$\prod \phi_i$: direct product of maps ϕ_i
$\cup_i\, \phi_i$: union of a chain of maps
$\mathfrak{e} : 0 \to A \to B \to C \to 0$: exact sequence

Groups, rings

0: number 0, element 0, or subgroup $\{0\}$
$\mathbb{N}$: set of positive integers
$\mathbb{Z}$: group of integers
$\mathbb{Q}$: group of rational numbers
$\mathbb{R}$: group of real numbers
$\mathbb{C}$: group of complex numbers
$\mathbb{T} \cong \mathbb{R}/\mathbb{Z}$: group of complex numbers of absolute value 1
$\mathbb{Z}(n)$: cyclic group of order n
$\mathbb{Z}(p^\infty)$: quasi-cyclic p-group
$\mathbb{Z}_{(p)}$: group or ring of rational numbers with denominators coprime to p
$\mathbb{Q}^{(p)}$: group or ring of rational numbers whose denominators are powers of p
J_p: group or ring of p-adic integers
$\mathbb{Q}_p^*$: group or field of p-adic numbers
H_σ: generalized Prüfer group of length σ

$N_\sigma(p)$: Nunke group of length σ for prime p
P_β: Walker group of length β
$A, B, C, G, H, T, \ldots$: groups
A_p: p-component of group A
$A_{(p)}$: localization of A at prime p
$E(A)$: injective hull of A
$\tilde{A}$: $\mathbb{Z}$-adic or p-adic completion of A
$\hat{A}$: Pontryagin dual of A
$\bar{B}$: torsion-complete group with basic subgroup B
$A^\bullet$: cotorsion hull of A
$A(\mathbf{t})$: subgroup of elements of types $\geq \mathbf{t}$
$A^*(\mathbf{t})$: subgroup generated by types $> \mathbf{t}$
$A[\mathbf{t}], A^*[\mathbf{t}]$: subgroups defined in terms of $\mathbf{t}$
$\mathfrak{G}, \mathfrak{H}$: non-commutative groups
$\mathfrak{z}(S)$: centralizer of set S in a group
R, S, $\ldots$: rings
R^+: additive group of ring R
$U(\mathsf{R})$: group of units of ring R
$\mathsf{F}^\times$: multiplicative group of field F
J: Jacobson radical of the ring
${}_{\mathsf{R}}M$: left module M over R

Special Notation

gcd: greatest common divisor
lcm: least common multiple
rk A: rank of A
$\mathrm{rk}_0\, A$, p-rk A: torsion-free rank, p-rank of A
$\mathrm{rk}^p A$: p-corank of A
fin rk A: final rank of p-group A
supp: support of a vector
bpd(A): balanced projective dimension of A
Type(A): type set of torsion-free finite rank group A
IT(A), OT(A): inner, outer type of torsion-free group A

Symbols

$\cong$: isomorphism
$\approx$: near-isomorphism
$\sim$: quasi-isomorphism
$\cong_{\mathcal{C}}$: isomorphism in category $\mathcal{C}$

$o(a)$: order of element a
$n|a$: integer n divides group element a
$h_p(a)$: (transfinite) height of element a at prime p
$\chi(a)$: characteristic of element a in torsion-free A
$A(\chi) = \{a \in A \mid \chi(a) \geq \chi\}$
$\mathbf{t}(a)$: type of element a
$B \leq A, B < A$: B is a (proper) subgroup of A
$\lhd$: normal subgroup
$B + C, \sum B_i$: subgroup generated by subgroups B, C, by subgroups B_i
$\langle S \rangle$: subgroup generated by S
$\langle S \rangle_*$: pure subgroup generated by S in a torsion-free group
$|A : B|$: index of subgroup B in A
$t(A) = tA$: torsion subgroup of A
$nA = \{na \mid a \in A\}$ where $n \in \mathbb{N}$
$A[n] = \{a \in A \mid na = 0\}$ where $n \in \mathbb{N}$
$n^{-1}B = \{a \in A \mid na \in B\}$ for $B < A$
C^-: $\mathbb{Z}$-adic closure of subgroup C in the given group
$p^\sigma A$: set of elements in A of p-height $\geq \sigma$
$A^1 = \cap_{n \in \mathbb{N}} nA$: first Ulm subgroup of A
A^σ: σth Ulm subgroup of A
$A_\sigma = A^\sigma / A^{\sigma+1}$: σth Ulm factor of A
$\underline{u} = (\sigma_0, \sigma_1, \ldots, \sigma_n, \ldots)$: indicator, increasing sequence of ordinals and ∞
$A(\underline{u})$: fully invariant subgroup of A defined by $\underline{u}$
$A \| B$: A is compatible with B
$\mathbf{L}(A)$: lattice of subgroups in A
$\mathfrak{T}$: lattice of all types
$\mathbf{t}, \mathbf{s}, \ldots$: types in torsion-free groups
$\mathbf{t}(A)$: type of the torsion-free group A
$\ell(A)$: length of p-group A
$f_\sigma(A) = p^\sigma A[p] / p^{\sigma+1} A[p]$: σth UK-invariant of p-group A
$f_\sigma(A, G)$: σth Hill invariant of A relative to subgroup G
$\mathrm{rk} A_{\mathbf{t}} = \mathrm{rk}(A(\mathbf{t}) / A^*(\mathbf{t}))$: Baer invariant for $\mathbf{t}$
$\mathfrak{A} = \{A_i \ (i \in I); \ \pi_i^j\}$: direct (inverse) system
$\mathbb{M}$, $\|a_{ik}\|$: matrix
$\mathbb{H}(a) = \|\sigma_{ik}\|$: height matrix of a in mixed group
$A(\mathbb{H}) = \{a \in A \mid \mathbb{H}(a) \geq \mathbb{H}\}$

Operations

$A \oplus C$: direct sum of A and C
$\oplus_{i \in I} A_i, \prod_{i \in I} A_i$: direct sum, direct product of the A_i with i changing over I
$\oplus_\kappa A, A^{(\kappa)}$: direct sum of κ copies of A

$\prod_\kappa A, , A^\kappa$: direct product of κ copies of A
$\prod_{i\in I}^{<\kappa} A_i$: κ-product of A_i
$\prod_{\mathsf{K}} A_i$: K-product (K=ideal in power-set of I)
$A^{(\mathbf{B})}$: Boolean power of A
$\mathrm{Hom}(A, C)$: group of homomorphisms from A to C
$\mathrm{Hom}_s(A, C)$: group of small homomorphisms
$\mathbb{Q}\,\mathrm{Hom}(A, C)$: group of quasi-homomorphisms
$\mathrm{Hom}_{\mathsf{R}}(A, C)$: group of R-homomorphisms between R-modules ${}_{\mathsf{R}}A, {}_{\mathsf{R}}C$
$\mathrm{End}\,A$: endomorphism ring (group) of A
$\mathrm{End}_s\,A$: ring of small endomorphisms of A
$\mathbb{Q}\,\mathrm{End}\,A$: quasi-endomorphism ring of torsion-free A
$\mathrm{Aut}\,A$: automorphism group of A
$A \otimes C$: tensor product of A and C
$A \otimes_{\mathsf{R}} C$: tensor product of A_{R} and ${}_{\mathsf{R}}C$ over R
$\mathrm{Tor}(C, A)$: torsion product of C and A
$\mathrm{Ext}(C, A)$: group of extensions of A by C
$\mathrm{Pext}(C, A)$: group of pure extensions of A by C
$\mathrm{Bext}(C, A), \mathrm{PBext}(C, A)$: group of (pre)balanced extensions of A by C
$\mathrm{Char}\,A$: character group of A in Pontryagin duality
$\mathrm{Mult}\,A$: group of ring multiplications on A
$\mathrm{Tr}_A(G)$: trace of A in G
$\varinjlim, \varprojlim$: direct, inverse limit
$\varprojlim^1$: first derived functor of inverse limit