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Asymptotic Methods in Mechanics of Solids

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Preface

The book is based on the special courses "Introduction to the asymptotic methods" and "Asymptotic method in mechanics" for postgraduate students at St. Petersburg State University and first read by Prof. P.E. Tovstik more than 40 years ago. The authors would like to underline the special role of Prof. P.E. Tovstik, who initiated the study of asymptotic methods applied to problems of solid mechanics at Saint-Petersburg (then Leningrad) State University and who is a teacher of the contributors.

The present book is a result of the scientific cooperation of researchers from the Departments of Theoretical and Applied Mechanics of the Faculty of Mathematics and Mechanics of St. Petersburg State University and the Department of Mathematics of the University of Ottawa.

Since in most of the papers in the collection on mechanics of solids published in 1993 [10] asymptotic ideas and methods were used the publisher proposed to supply the volume with survey by S.M. Bauer, S.B. Filippov, A.L. Smirnov, and P.E. Tovstik entitled "Asymptotic Methods in Mechanics with Applications to Thin Shells and Plate." Later this survey encouraged the authors to write a textbook on the application of the asymptotic method in mechanics. The present book is the elaborated version of the Russian edition published in 2007. The book is supplied with the Introduction containing a brief discussion of publications on asymptotic methods in mechanics of solids, especially those that are not referred to in the main text. The reference section is significantly enlarged.

The authors believe that studying the basics of asymptotic methods may be useful to advanced undergraduate, postgraduate, and Ph.D. students in Mathematics, Physics, and Engineering, to researchers and engineers working in the analysis and construction of thin-walled structures and continuous media, and to applied mathematicians who are interested in asymptotic methods in problems of mechanics.

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Contents

Introduction

Asymptotic methods of various types have been successfully used since almost the birth of science itself. Transformation of the ideas of asymptotic analysis to a specific area in mathematics happened at the end of the nineteenth century when Henri Poincare introduced the idea of the asymptotic series and gave a rigorous definition of an asymptotic expansion. In the twentieth century asymptotic methods were widely used in different areas of applied mathematics. Now asymptotic methods based on the expansion of solutions in series in small or large parameters or coordinates hold a central place among approximate methods. Asymptotic methods give a qualitative characteristic of the behavior of solutions. Besides that, in some cases, asymptotic expansions have small errors for a rather wide parameter domain.

The number of textbooks, monographs, and journal papers devoted to the asymptotic methods is rather large and it grows constantly. The asymptotic expansions are also discussed in publications on general methods of solution of applied problems. For example, the book by Bender and Orszag [11] contains many interesting examples of application of the perturbation methods.

For introduction to the general principles of asymptotic analysis, the textbooks by Nayfeh [49, 50] may be recommended. In these books the definitions of the asymptotic series and simple operations with them are introduced together with methods of solution of algebraic and transcendent equations, methods of integrations (Laplace method, stationary phase method, and steepest descent method), and classical methods of solution of linear and nonlinear ordinary differential equations with a parameter, including the multiscale method. Some of these problems are discussed in detail in books by de Bruijn [14], Erdèlyi [20], Kevorkian and Cole [37], and Holms [34]. In addition, monograph [34] includes chapters on homogenization, discrete equations, wave propagation, and Lyapunov-Schmidt method. Books by Maslov [43], Maslov, and Nazaikinskii [44] contain descriptions of the most general methods of asymptotic integration of linear and nonlinear non-stationary partial differential equations.

In 1963, Martin David Kruskal coined the term Asymptology to describe the “art of dealing with applied mathematical systems in limiting cases.” He tried to show that asymptology is a special branch of knowledge, intermediate, in some sense, between science and art. Kruskal's ideas were lately developed in works by Andrianov and Manevitch [3] and Barantsev [8], which contained the heuristic description of different asymptotic methods.

In the majority of cases in the listed books the authors limit themselves with construction of a few first terms of the asymptotic series without rigorous estimating the errors. These are so-called formal asymptotic expansions. The estimates for the asymptotic expansions are given by Murdock [48], Fröman and Fröman [27], Fedoruk [23], Evgrafov [21].

One of the main areas of application of asymptotic methods is the analysis of differential equations. In asymptotic integration of differential equations containing small parameter the cases of regular and singular perturbations are considered separately [33]. The perturbation is called regular if the orders of the differential equation or the system of equations do not change when the small parameter becomes equal to zero. For singular perturbation, when the small parameter is set equal to zero the order of the equation or system decreases since the small parameter is a multiplier at the higher derivatives. Asymptotic expansions of solutions of singularly perturbed equations are usually divergent series [50].

In the presented book special attention is devoted to the analysis of singular perturbed differential equations. The authors of this book use different asymptotic methods to solve applied problems not pretending to develop the general theory of singular perturbations. The systematic studies of asymptotic solutions of some singular problems may be found in monographs by Eckhaus [19] and Lomov [42].

Consider singularly perturbed linear differential equation of the nth order

$$\sum_{k=0}^{n} \mu^k a_k(x)\frac{d^k y}{dx^k} = 0, \tag{1}$$

where $\mu > 0$ is a small parameter. Solution of (1) we seek in the form

$$y(x,\mu) = \sum_{k=0}^{\infty} \mu^k u_k(x)\exp\left(\frac{1}{\mu}\int_{x_0}^{x} \lambda(x)dx\right). \tag{2}$$

Substituting (2) into (1) and equating the coefficients at μ^k to zero we get the system of equations to find the unknown functions $\lambda(x)$ and $u_k(x)$. For nontrivial solutions $\lambda(x)$ is a root of the characteristic equation

$$\sum_{k=0}^{n} a_k(x)\lambda^k = 0 \tag{3}$$

For $a_n(x) \neq 0$ Eq. (3) has n roots. Let $\lambda(x)$ be a simple root of Eq. (3). Then series (2) may be constructed with the coefficients $u_k(x)$, which makes Eq. (1) an identity. Such series is called formal asymptotic solution. Further, we limit ourselves to construction of such solutions leaving aside the question of existence of exact solutions, for which the obtained solutions are asymptotic expansions.

For the system of singularly perturbed linear differential equations

$$\mu \frac{d\boldsymbol{y}}{dx} = \boldsymbol{A}(x)\boldsymbol{y}, \quad \mu > 0, \tag{4}$$

where $\boldsymbol{y}$ is the nth dimensional vector and $\boldsymbol{A}$ is the square matrix of the nth order, for formal asymptotic solution we seek the form

$$\mathbf{y}(x,\mu) \simeq \sum_{k=0}^{\infty} \boldsymbol{U}_k(x)\mu^k \exp\left(\frac{1}{\mu}\int_{x_0}^{x} \lambda(x)dx\right).$$

The function $\lambda(x)$ satisfies the characteristic equation

$$\det(\boldsymbol{A}_0(x) - \lambda(x)\boldsymbol{I}_n) = 0,$$

where $\boldsymbol{I}_n$ is the identity matrix of the order n.

If all n roots of the characteristic equation are simple, then we get n linearly independent asymptotic solutions of Eq. 1 or system (4), which may be used to solve the boundary value problems. Solutions, for which $\Re(\lambda) \neq 0$, increase or decrease exponentially are called the edge effect integrals in solid mechanics and boundary layer integrals in hydromechanics. For $\Re(\lambda) = 0$ the solution rapidly oscillates and for $\lambda = 0$ the solution changes slowly.

The difficulties arise when the characteristic equation has multiple roots. First consider the case of the zero root of the multiplicity m that is often met in applications. The linear differential equation of the order $n = l + m$

$$L_\mu y = \sum_{k=0}^{l} \mu^k a_{k+m}(x)\frac{d^{k+m}y}{dx^{k+m}} + \sum_{k=0}^{m-1} a_k(x)\frac{d^k y}{dx^k} = 0 \tag{5}$$

for $\mu = 0$ transforms to the equation of the order m

$$L_0 y = \sum_{k=0}^{m} a_k(x)\frac{d^k y}{dx^k} = 0. \tag{6}$$

If Eq. (5) is multiplied by μ^m, we get it in the form (1), for which the characteristic equation

$$\sum_{k=0}^{l} a_{k+m}(x)\lambda^{k+m} = 0$$

has zero root of multiplicity m. Let $a_n(x) \neq 0$, $a_m(x) \neq 0$ and all roots of equation

$$\sum_{k=0}^{l} a_{k+m}(x)\lambda^k = 0$$

are simple. Then Eq. (5) has l solutions of form (2). The remaining m solutions are slowly changing functions in x and have the expansions

$$y(x,\mu) = \sum_{k=0}^{\infty} \mu^k v_k(x), \tag{7}$$

If the multiplicity of the roots changes with the argument x, then the points at which the changes happen are called the turning points. The first approximate studies of the behavior of solutions at the neighborhoods of the turning points were made by Wentzel et al. [11, 37], thus the methods of integration of equations with the turning point are sometimes called WKB-methods. The constructions of asymptotic solutions for system of differential equation of the second and higher orders under different assumptions on the character of the turning point are made in the fundamental monograph by Wasow [65]. Usually, for construction of the asymptotic solutions for equations with turning point the method of comparison equations is used [64]. These equations have the same singularities as the initial equations, but they are simpler than the last ones.

The equation of the second order

$$\mu^2 \frac{d^2y}{dx^2} - q(x)y = 0$$

with the small parameter at the derivative has the turning point $x = x_*$, if $q(x_*) = 0$. For the simple turning point, for which $q'(x_*) \neq 0$, the asymptotic expansions of the solutions may be expressed in the Airy functions $\mathrm{Ai}(\eta)$ and $\mathrm{Bi}(\eta)$. These functions are the solutions of the comparison equation

$$\frac{d^2v}{d\eta^2} - \eta v = 0.$$

In the general formulation the problem of asymptotic integration of equations with the turning points has not been solved yet. Only some special cases for the equations encountered in applications has been analyzed.

In the paper by Lin and Rabestein [41] the fourth order equation describing the stability of the laminar viscous flow is considered. Its characteristic has the form $\lambda^4 + x\lambda^2 = 0$, and its roots are quadruple for $x = 0$. The analysis of axisymmetric vibrations of noncylindrical shell of revolution may be reduced to study of the sixth order equation with the characteristic equation $\lambda^6 + f(x)\lambda^2 = 0$ and sextuple turning point. The asymptotic solutions for that equation were constructed by Goldenveizer et al. [30] with the help of the comparison equations method.

The solution of many problems in mechanics of solids may be reduced to solution of the boundary value problems for linear differential equations. The approximate solution of the boundary value problem may be obtained by substituting the asymptotic expansions of solutions into the boundary conditions. For singular perturbed ordinary differential equations such expansion may have the form (2). In this case the method to obtain the approximate solution of the boundary value problem depends on behavior of integrals (2), which is defined, in turn, by the values of the roots of the characteristic equation.

Consider boundary value problem for Eq. (5). Assume that its solution satisfies n homogeneous boundary conditions. When the small parameter μ vanishes Eq. (5) degenerates to Eq. (6) which has the order of $m < n$. Therefore, solution of (6) cannot satisfy all n boundary conditions of the initial boundary value problem. The questions are: (i) can the solution of unperturbed Eq. (6) be zero approximation to solution of the initial problem and (ii) which m out of n given boundary conditions should be selected for Eq. (6)? The answer to these questions is given in the classical paper by Vishik and Lyusternik [62]. In that paper the concept of regular degeneracy, for which the solution of the boundary value problem converges to solution of the unperturbed problem as $\mu \to 0$, is given.

For regular degeneracy it is necessary that all solutions of form (2) be the edge effect integrals and the number of decreasing and increasing integrals corresponds to the number of the boundary conditions on the left and right ends of the integration interval. In this case the solution of the initial boundary value problem is represented as the sum of solutions (2) and (7). In problems of mechanics of solids solution of form (7) is called the main state, the boundary conditions for the unperturbed problem are the main boundary conditions, and the other conditions are the auxiliary conditions.

The thin shell theory provides numerous problems to be solved by asymptotic methods. Two-dimensional differential equations of the eighth order in the theory of shells are singularly perturbed since they contain the natural small parameter h, the dimensionless relative thickness of the shell, which is a factor at the higher derivatives. Often the dimensionless parameter μ, which is proportional to $\sqrt{h}$ is used instead of h. For $\mu = 0$ we have the unperturbed system of equations of the fourth order, which is called membrane (or momentless).

The foundations of asymptotic analysis of linear equations of the theory of shells are formulated in the classical works by Gol'denveizer [28, 29]. In those works an important concept of the index of variation of solution is introduced. The index of variation for function F is a real number t, such that

$$\frac{\partial F}{\partial x} \sim \mu^{-t} F, \quad \text{as} \quad \mu \to 0.$$

For $t > 0$, the function F varies fast while for $t \leq 0$ it varies slowly. Solutions (2) have the index of variation $t = 1$, at the same time the index of variation for solutions (7) is zero.

The analysis of possible solutions of shell equilibrium equations with different indices of variations in two space variables given in [29] permits to classify the main stress states: membrane (momentless) state, the edge effect, etc. Based on this analysis the approximate methods of solution of the problems of shell statics have been developed. For example, under some conditions the solution may be sought in the form of a sum of the main membrane state and the edge effect integrals.

For the shells closed in the circumferential direction it is convenient to select as space coordinates on the shell mid-surface (neutral surface) the length of the meridian arc s and the angle in the circumferential direction φ. After separating the variables

$$y(s, \varphi) = y(s)e^{im\varphi}, \tag{8}$$

the equations of shell statics transform to ordinary differential equations, the coefficients of which depend on the wavenumber in the circumferential direction m. In equations describing vibrations or buckling of shells the additional dimensionless parameter Λ, which is proportional to the square of the natural frequency or critical loading, appears. The form of the asymptotic solutions depends on the relations between parameters μ, m and Λ.

For $m = 0$ the shell deformation is axisymmetric and it is described by the system of differential equations of the sixth order. For low frequency vibrations of cylindrical and conical shells, which are of the great importance to the applications, $m \sim \mu^{-1/2}$ and $\Lambda \sim \mu^2$. In this case the degeneration of the initial system of the eighth order to the system of the fourth order is regular. The stress state described by the unperturbed system is called semi-momentless. For the cylindrical shell the unperturbed system has an explicit solution. The detailed asymptotic analysis of free vibrations of shells is given by Goldenveizer et al. [30].

To solve the linear problems of buckling of momentless initial stress state, the same methods of asymptotic integration as for the problems of free vibrations are used. The only difference is that for buckling problems the lowest eigenvalue corresponding to the critical load is sought as a rule. Numerous methods and results on buckling of shells are included in the book by Tovstik and Smirnov [56]. In this book the main attention is devoted to the methods of construction of asymptotic expansions of localized buckling modes based on algorithm proposed by Maslov [43]. In the classical problems of shell buckling the radii of the curvature of the mid-surface, its thickness, and momentless initial stress resultants are usually constant. In this case the pits cover the entire surface of a shell under buckling. On the other hand, if the parameters of the shell and the initial stress state depend on the space coordinates, then the localization of the buckling pits may happen at the vicinities of some lines or points on the mid-surface, which are called the weakest lines (points).

In the book by Tovstik and Smirnov [56] the buckling modes for the convex shells of revolution localized at the neighborhood of the weakest parallel have been constructed. Under nonhomogeneous axial compression of cylindrical shells the buckling modes are localized at the vicinity of the weakest generatrix. The convex shell and cylindrical shell may buckle under nonhomogeneous compression with the buckling mode localized at the weakest point. For these cases the asymptotic expansions for the buckling modes are found. The method of asymptotic separation of the variables is developed and applied to represent the total stress state of the shell as a sum of the semi-momentless state and edge effect. Simultaneously, the problem of separation of the boundary conditions at the shell edges for the main and auxiliary conditions is solved. This method is applied for cylindrical and conic shells, for which the problem may not be reduced to one dimensional by separating the variables in form (8).

In the book by Filippov [26] the method of asymptotic separation of the variables is applied for analysis of free vibrations and buckling under external pressure of the joint shells and shells reinforced by rings. For cylindrical and conic shells the boundary conditions on the shells joint lines and on the lines of the contact of the shell and rings are split into the main and auxiliary boundary conditions.

The solutions of the problems of shell theory by means of the Lyapunov-Schmidt procedure, multiscale method, homogenization, Padé approximants, and other asymptotic methods are included in the monograph by Andrianov et al. [2], which contains a vast bibliography.

The theoretical results in this book are supplemented with the analysis of problems and exercises. In the solution of many problems, asymptotic and numerical methods are used together. The combination of these two methods makes the results more reliable, permits to estimate the applicability domain for asymptotic formulas, and makes easier the numerical analysis of the problem. For example, when evaluating a root of an equation one should know the interval boundary for the root. This boundary may be found by means of asymptotic methods.

Asymptotic estimates of functions, solutions of algebraic and transcendental equations, and also systems of linear algebraic equations are considered in the first chapter. This part is traditional for many manuals on asymptotic methods. However, some of the questions are rarely discussed in textbooks. For example, the Newton polyhedron, which is a generalization of the Newton polygon for equations with two or more parameters, is considered in Chap. 1. Then the important concept of the index of variation for functions is introduced. Special attention is devoted to eigenvalue problems containing a small parameter.

Chapter 2 is dedicated to asymptotic methods for calculating integrals (integration by parts, Laplace transform, stationary phase, saddle point) which are used later in the book to construct asymptotic expansions for solutions of differential equations containing small parameters.

In Chap. 3 the construction of solutions of regularly perturbed ordinary differential equations is discussed. The traditional methods include Poincaré's averaging and multiscale methods. In addition, linear boundary value problems for differential equations with small parameters are considered. Problems for equations with fast oscillating coefficients are also analyzed.

The main part of the book is Chaps. 4 and 5, which deal with methods of asymptotic solutions of linear singularly perturbed boundary value and eigenvalue problems without or with turning points, respectively. In Chap. 4, the asymptotic expansions of linearly independent solutions of systems of linear ordinary differential equations with small parameters at the derivatives are constructed. These asymptotic expansions are later used in the book for approximating solution of nonhomogeneous boundary value and eigenvalue problems. The cases where the eigenfunctions are localized near the edge of the integration interval are also studied. As examples, one-dimensional equilibrium, dynamics, and stability problems for rigid bodies and solids are examined.

In Chap. 5, the singular perturbed problems are analyzed in the case where there exist turning points inside the interval of integration. At a turning point, the asymptotic expansions obtained in Chap. 4 are not valid, since in the expressions one of the functions in the denominator is zero. Approximate asymptotic solutions in a neighborhood of a turning point for linear differential equations of the second order with small parameters at the higher derivative are given in Chap. 5. Then the eigenvalue problems describing the vibration of circular plates and shells of revolution are examined. Asymptotic expansions for the eigenfunctions localized near the internal point of the interval of integration are also found.

Finally, in Chap. 6 the asymptotic integration of nonlinear differential equations is considered, where questions of singular perturbation and ramification of solutions are discussed.

Many of the problems of asymptotic integration are not discussed in this book. Among them there is the method of matching of asymptotic expansions, which is widely used in hydromechanics. Its description may be found, for example, in books by Van Dyke [59] and Hinch [33]. One of the versions of this method is the application of Padé approximants, numerous examples of which are given in the monograph by Baker and Graves-Morris [6].

In our book we analyze only stationary vibrations. We note that the considerable progress has been also made in study of the process of wave propagations by asymptotic methods. In the book by Mikhasev and Tovstik [47] the authors study both localized buckling modes and the motion of the wave packages running on the shell of revolution either in circumferential on axial directions. The book by Babich and Buldyrev [5] concerns the analysis of short-wave asymptotics for solution of Helmholtz equation for the wave propagation with constant or variable wave speed in two-dimensional or three-dimensional spaces. In the book by Kaplunov et al. [36] the asymptotic approach is used to describe the waves of different types in thin elastic solids and in particularly in shells of revolution.

One of the important areas of application of asymptotic methods, which is not included in this book due to its complexity, is the continuum mechanics in the narrow domains. These are the problems of thin-walled beams, plates, and shells theory and also the contact problems for solids on different dimensions. The asymptotic methods are applicable here since these problems contain the geometric small parameter, ratio of the minimum and maximal solid dimensions. In applications the equations are simplified as usual as a result of assumptions on distribution of the unknown functions in the thickness direction. In this case one of the goals of the asymptotic analysis is the verification of the hypotheses and the estimate of the errors happened under the assumptions.

The monograph by Nazarov [51] concerns these problems. It contains the asymptotic expansions of solutions of static problems and problems of free vibrations of beam, plates, and shells. The main attention is devoted to the error estimates, when only the main terms of the asymptotic expansions are considered and to the solvability of the boundary value problems, which appear in the process of the asymptotic integration.

In the book by Gol'denveizer [28] the method to derive the equations of the theory of shells from the three-dimensional equations of the theory of elasticity is proposed. The asymptotic approach developed by Kaplunov et al. in [36] is a dynamical generalization of Goldenveizer's method of asymptotic integration of partial differential equations in narrow domains. In the monograph by Ciarlet [16] two-dimensional equations of the theory of shells describing the membrane stress state are derived by means of the asymptotic method and strong error estimates are obtained. An ingenious sequence of the shell theories refining one another is given by Libai and Simmonds [40].

The authors do not claim the bibliography section to be complete. The references include mostly textbooks and monographs concerning the methods and problems considered in the book.