

SpringerBriefs in Mathematics

Series editors

Nicola Bellomo, Torino, Italy
Michele Benzi, Atlanta, USA
Palle E.T. Jorgensen, Iowa City, USA
Tatsien Li, Shanghai, China
Roderick Melnik, Waterloo, Canada
Otmar Scherzer, Vienna, Austria
Benjamin Steinberg, New York, USA
Lothar Reichel, Kent, USA
Yuri Tschinkel, New York, USA
G. George Yin, Detroit, USA
Ping Zhang, Kalamazoo, USA

SpringerBriefs in Mathematics showcases expositions in all areas of mathematics and applied mathematics. Manuscripts presenting new results or a single new result in a classical field, new field, or an emerging topic, applications, or bridges between new results and already published works, are encouraged. The series is intended for mathematicians and applied mathematicians.

More information about this series at <http://www.springer.com/series/10030>

Simon Hubbert · Quốc Thông Lê Gia
Tanya M. Morton

Spherical Radial Basis Functions, Theory and Applications

 Springer

Simon Hubbert
School of Economics, Mathematics
and Statistics
Birkbeck, University of London
London
UK

Tanya M. Morton
MathWorks
Cambridge
UK

Quốc Thông Lê Gia
School of Mathematics
The University of New South Wales
Sydney, NSW
Australia

ISSN 2191-8198

ISSN 2191-8201 (electronic)

SpringerBriefs in Mathematics

ISBN 978-3-319-17938-4

ISBN 978-3-319-17939-1 (eBook)

DOI 10.1007/978-3-319-17939-1

Library of Congress Control Number: 2015937749

Mathematical Subject Classification: 33C55, 35R01, 35Q86, 41A05, 41A24, 42A16, 42B37, 65D05, 65D15

Springer Cham Heidelberg New York Dordrecht London

© The Author(s) 2015

This work is subject to copyright. All rights are reserved by the Publisher, whether the whole or part of the material is concerned, specifically the rights of translation, reprinting, reuse of illustrations, recitation, broadcasting, reproduction on microfilms or in any other physical way, and transmission or information storage and retrieval, electronic adaptation, computer software, or by similar or dissimilar methodology now known or hereafter developed.

The use of general descriptive names, registered names, trademarks, service marks, etc. in this publication does not imply, even in the absence of a specific statement, that such names are exempt from the relevant protective laws and regulations and therefore free for general use.

The publisher, the authors and the editors are safe to assume that the advice and information in this book are believed to be true and accurate at the date of publication. Neither the publisher nor the authors or the editors give a warranty, express or implied, with respect to the material contained herein or for any errors or omissions that may have been made.

Printed on acid-free paper

Springer International Publishing AG Switzerland is part of Springer Science+Business Media
(www.springer.com)

*Simon dedicates this book to his partner
Michelle and his two daughters Nancy
and Clara.*

*Thong dedicates this book to his wife Tram
Linh.*

*Tanya dedicates this book to her husband
Chris and her two sons Kai and Eldin.*

Preface

In recent years mathematicians and researchers within the approximation theory community have become increasingly interested in using tools from approximation theory to develop numerical methods for problems set on spheres. Given that the two-dimensional sphere serves as a model for the surface of the earth, these problems are often generated by real-world applications. For instance, as more and more satellites are launched into space, the acquisition of global data is becoming more widespread and hence there is now increased demand for spherical data processing solutions. Another important application is in the quest to improve weather forecasting models through the development of efficient algorithms for solving partial differential equations (PDEs) posed on the surface of the sphere. There is also need for spherical approximation in areas other than geoscience and meteorology. For instance, one may want to construct a smooth surface that encloses a cloud of scattered points in Euclidean space, this can be achieved using spherical approximation solutions. This has numerous applications, for example, to model 3D objects for computer graphics, or to create a boundary model representing the safe operating envelope of internal combustion engines.

This book serves to provide an introduction to the theory and applications of spherical (radial) basis functions (SBFs), which represent one of the most promising emerging technologies for solving spherical problems. SBFs are closely related to the more famous family of radial basis functions (RBFs) which are already well-established tools for solving data fitting problems and PDEs over regions in Euclidean space. RBFs have a much longer history than SBFs and so, consequently, much more is known about them, indeed [Buh03; Fass07; Wen05] are three excellent textbooks devoted to their theoretical properties and their practical implementations. Our primary aim in this book is to present enough practical and theoretical details to enable the reader to implement SBF techniques to solve real problems and also, if desired, to pursue further theoretical studies in this exciting area. In Chap. 1 we set out our motivation for studying SBFs and provide the background tools from functional analysis which will be used throughout the book. In Chap. 2 we demonstrate how key ideas and concepts from the interpolation theory of RBFs in Euclidean space can be recast into the spherical setting and,

in doing so, we introduce the notion of SBFs and we show how they can be used to provide unique solutions (SBF interpolants) to data fitting problems on the sphere. Furthermore, we also reveal a simple variational framework for SBF interpolation and show how this can be used to analyse the accuracy of a particular SBF interpolant to a given target function. In Chap. 3 we pursue the error analysis in much greater detail. Specifically, we present the technical ingredients of an error bounding strategy which we then use to provide much improved error estimates for SBF interpolation. In Chap. 4 we test the theory and present the results of numerical experiments for the SBF method for solving data fitting problems on the sphere. In the final two chapters of the book we move away from data fitting applications and concentrate more on investigating how SBF approximations can be used to solve PDEs on spheres. In Chap. 5 we focus more on computational issues and propose a preconditioning strategy to speed up the iterative solution of an elliptic PDE. Finally, in Chap. 6, we examine the inhomogeneous heat equation as an example of a parabolic PDE. Here we develop a collocation solution method and provide a full error analysis when the time variable is discretized using either the backward Euler or the Crank-Nicolson method.

In summary, the material covered in this book is aimed at graduate students and researchers in mathematics and related fields such as the geophysical sciences and statistics. We have tried to make the exposition as clear and as self-contained as possible and have made efforts to ensure that technical details are explained in a friendly and readable style. We hope this will encourage the reader to delve deeper and discover more.

January 2015

Simon Hubbert
Quốc Thông Lê Gia
Tanya M. Morton

Contents

1	Motivation and Background Functional Analysis	1
1.1	Introduction	1
1.2	Notations	1
1.3	Motivation	3
1.4	Hilbert Space Theory	8
1.5	Spherical Harmonics and Fourier Analysis	10
1.6	Sobolev Spaces in Euclidean Space	14
1.7	Sobolev Spaces on the Unit Sphere	21
2	The Spherical Basis Function Method	29
2.1	Introduction	29
2.2	A Brief History of the RBF Method	30
2.3	The Spherical Basis Function Method	39
2.4	Framework for Pointwise Error Estimates	47
2.5	Pointwise Error Estimate I	50
2.6	Pointwise Error Estimate II	52
3	Error Bounds via Duchon’s Technique	59
3.1	Duchon’s Recipe for the Sphere	59
3.2	Global Error Bounds for SBF Interpolation	76
4	Radial Basis Functions for the Sphere	85
4.1	Duchon Splines for the Sphere	85
4.2	Numerical Investigation	91
5	Fast Iterative Solvers for PDEs on Spheres	97
5.1	Introduction	97
5.2	The Weak Formulation of the PDE	99
5.3	The Additive Schwarz Method	101

5.4	A Subspace Decomposition Algorithm	103
5.5	An Upper Bound for the Condition Number $\kappa(\mathcal{P})$	104
5.6	An Overlapping Additive Schwarz Algorithm	111
5.7	Numerical Results	113
6	Parabolic PDEs on Spheres.	121
6.1	Introduction	121
6.2	The Homogeneous Semi-discrete Problem	122
6.3	The Inhomogeneous Semi-discrete Problem	125
6.4	Time Discretization Using the Backward Euler Method.	128
6.5	Time Discretization Using the Crank-Nicolson Method	132
6.6	Numerical Experiments on S^2	135
	References.	139