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Advanced Functional Evolution Equations and Inclusions

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We dedicate this book to our family members. In particular, Saïd Abbas dedicates to the memory of his father Abdelkader Abbas; and Mouffak Benchohra makes his dedication to the memory of his father Yahia Benchohra

Preface

Functional differential equations and inclusions occur in a variety of areas of biological, physical, and engineering applications, and such equations have received much attention in recent years. This book is devoted to the existence of local and global mild solutions for some classes of functional differential evolution equations and inclusions, and other densely and non-densely defined functional differential equations and inclusions in separable Banach spaces or in Fréchet spaces. Some of these equations and inclusions present delay which may be finite, infinite, or state-dependent. Other equations are subject to impulses effect. The tools used include classical fixed point theorems and the measure of non-compactness (MNC). Each chapter concludes with a section devoted to notes and bibliographical remarks. All the presented abstract results are illustrated by examples.

The content of the book is new and complements the existing literature devoted to functional differential equations and inclusions. It is useful for researchers and graduate students for research, seminars, and advanced graduate courses, in pure and applied mathematics, engineering, biology, and all other applied sciences.

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Saïda, Algeria — Saïd Abbas
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Introduction

Nonlinear evolution equations, i.e., partial differential equations with time t as one of the independent variables, arise not only from many fields of mathematics, but also from other branches of science such as physics, mechanics, and material science. For example, Navier–Stokes and Euler equations from fluid mechanics, nonlinear reaction-diffusion equations from heat transfers and biological sciences, nonlinear Klein–Gordon equations and nonlinear Schrödinger equations from quantum mechanics, and Cahn–Hilliard equations from material science, to name just a few, are special examples of nonlinear evolution equations. See the books [174, 176–178].

Functional differential equations and inclusions arise in a variety of areas of biological, physical, and engineering applications, and such equations have received much attention in recent years. A good guide to the literature for functional differential equations is the books by Hale [131], Hale and Verduyn Lunel [133], Kolmanovskii and Myshkis [148], and the references therein. During the last decades, existence and uniqueness of mild, strong, classical, almost periodic, almost automorphic solutions of semi-linear functional differential equations and inclusions has been studied extensively by many authors using the semigroup theory, fixed point argument, degree theory, and measures of non-compactness. We mention, for instance, the books by Ahmed [16], Diagana [103], Engel and Nagel [106], Kamenskii et al. [144], Pazy [168], Wu [184], Zheng [187], and the references therein. In recent years, there has been a significant development in evolution equations and inclusions; see the monograph of Perestyuk et al. [169], the papers of Baliki and Benchohra [33, 37], Benchohra and Medjedj [55, 56], Benchohra et al. [82], and the references therein.

Neutral functional differential equations arise in many areas of applied mathematics and such equations have received much attention in recent years. A good guide to the literature for neutral functional differential equations is the books by Hale [131], Hale and Verduyn Lunel [133], Kolmanovskii and Myshkis [148], and the references therein. Hernandez in [137] proved the existence of mild, strong, and periodic solutions for neutral equations. Fu in [117, 118] studies the controllability on a bounded interval of a class of neutral functional differential equations. Fu and

Ezzinbi [119] considered the existence of mild and classical solutions for a class of neutral partial functional differential equations with nonlocal conditions. Various classes of partial functional and neutral functional differential equations with infinite delay are studied by Adimy et al. [10–12], Belmekki et al. [52], and Ezzinbi [108]. Henriquez [136] and Hernandez [137, 138] studied the existence and regularity of solutions to functional and neutral functional differential equations with unbounded delay. Balachandran and Dauer have considered various classes of first and second order semi-linear ordinary, functional and neutral functional differential equations on Banach spaces in [43]. By means of fixed point arguments, Benchohra et al. have studied various classes of functional differential equations and inclusions and proposed some controllability results in [28, 33, 33, 37, 58, 72, 73, 75, 76, 80]. See also the works by Gatsori [120], Li et al. [155], Li and Xue [156], and Li and Yong [157].

Impulsive differential equations and inclusions appear frequently in applications such as physics, aeronautic, economics, engineering, and population dynamics; see the monographs of Bainov and Simeonov [39, 40], Benchohra et al. [81], Erbe and Krawcewicz [107], Graef et al. [127], Samoilenko and Perestyuk [172], and Perestyuk et al. [169], and the paper of Coldbeter et al. [95] where numerous properties of their solutions are studied. In this way, they make changes of states at certain moments of time between intervals of continuous evolution such changes can be reasonably well approximated as being instantaneous changes of this state which we will represent by impulses and then these processes are modeled by impulsive differential equations and for this reason the study of this type of equations has received great attention in the last years. There has been a significant development in impulsive theory especially in the area of impulsive differential equations with fixed moments. See, for instance, the monographs by Benchohra et al. [81], Lakshmikantham et al. [150], and Samoilenko and Perestyuk [172]. There exists an extensive literature devoted to the case where the impulses are absent (i.e., $I_k = 0, k = 1, \dots, m$), see, for instance, the monograph by Liang and Xiao [158] and the paper by Schumacher [158]. We mention here also the use of impulsive differential equations in the study of oscillation and non-oscillation of impulsive dynamic equations, see, for instance, the papers of Graef et al. [124, 125], oscillation of dynamic equations with delay was considered in [13, 14]. During the last 10 years impulsive ordinary differential inclusions and functional differential equations and inclusions have attracted the attention of many mathematicians and are intensively studied. At present the foundations of the general theory and such kind of problems are already laid and many of them are investigated in detail in [58, 59, 63, 79, 81, 107] and the references therein.

It is well known that the issue of controllability plays an important role in control theory and engineering because they have close connections to pole assignment, structural decomposition, quadratic optimal control, observer design, etc. In recent years, the problem of controllability for various kinds of differential and impulsive differential systems has been extensively studied by many authors [71, 155–157, 186] using different approaches. Several authors have extended the controllability concept to infinite dimensional systems in Banach space with unbounded operators, see the monographs [85, 98, 157, 186] and the references therein. Sufficient conditions for controllability are established by Lasiecka and Triggiani [153]. Fu in [117, 118] studied the controllability on a bounded interval of a class of neutral functional differential equations. Fu and Ezzinbi [119] considered the existence of mild and classical solutions for a class of neutral partial functional differential equations with nonlocal conditions. Adimy et al. [10–12] studied some classes of partial functional and neutral functional differential equations with infinite delay. When the delay is infinite, the notion of the phase space $\mathcal{B}$ plays an important role in the study of both qualitative and quantitative theory. A usual choice is a semi-normed space satisfying suitable axioms, which was introduced by Hale and Kato in [132], see also Corduneanu and Lakshmikantham [97] and Kappel and Schappacher [145].

The literature related to ordinary and partial functional differential equations with delay is very extensive. On the other hand, functional differential equations with state-dependent delay appear frequently in applications as model of equations, and for this reason the study of this type of equations has received great attention in the last year, see, for instance [31, 183] and the references therein. The literature related to partial functional differential equations with state-dependent delay is limited; see [139, 171].

Several authors have considered extensively the problem

$$x'(t) = A(t)x(t) + f(t, x_t)$$

when $A(t) = A$. Existence of mild solutions is developed by Heikkila and Lakshmikantham [134], Kamenski et al. [144], and the pioneer Hino and Murakami paper [141] for some semi-linear functional differential equations with finite delay. By means of fixed point arguments, Benchohra and his collaborators have studied many classes of first and second order functional differential inclusions on a bounded interval with local and nonlocal conditions in [59, 60, 62, 64, 65, 77, 78, 121]. Extension to the semi-infinite interval is given by Benchohra and Ntouyas in [58, 61]. When A depends on time, Arara et al. [26, 28] considered a control multi-valued problem on a bounded interval. Uniqueness results of mild solutions for some classes of partial functional and neutral functional differential evolution equations on the semi-infinite interval $J = \mathbb{R}_+$ for a finite delay with local and nonlocal conditions were given in [33, 37]. When the delay is infinite, existence and uniqueness results for evolution problems are proposed in [33], and controllability result of mild solutions for the evolution equations are given in [15, 36]. The case when A is non-densely defined and generates an integrated semigroup was done by Benchohra et al. [80]. Some global existence results for impulsive differential equations and inclusions were obtained by Guo [129], Graef and Ouahab [126], and the references therein.

Partial functional evolution equations and inclusions with infinite and state-dependent delay, controllability on finite interval are our concerns. Our approach is based upon the fixed point theory for multi-valued condensing maps under assumptions expressed in terms of the MNC [144].

In the last three decades, the theory of C_0-semigroup has been developed extensively, and the achieved results have found many applications in the theory of partial differential equations, for instance see [106, 122, 168] and the papers of Arara et al. [26, 27] and Benchohra et al. [74]. Recently, increasing interest has been observed in applications to impulsive differential equations and inclusions, see Liu [69, 159]. The case where the generator of the semigroup is non-densely defined, the existence of integral solutions on compact intervals for differential equations and inclusions were studied by Adimy et al. [8–10], Arendt [29, 30], Ezzinbi and Liu [109, 110], and Henderson and Ouahab [135]. The model with multi-valued jump sizes arises in a control problem where we want to control the jump sizes in order to achieve given objectives. There are very few results for impulsive evolution inclusions with multi-valued jump operator, see [161]. We present the existence of solutions for both densely or non-densely defined impulsive functional differential inclusions.

The multi-valued jumps (i.e., the difference operator $\Delta x\big|_{t=t_k} \in \mathcal{I}_k(x(t_k^-)))$ is a natural model of an impulsive system where the jump sizes are not deterministic as in [17–19, 161] but rather they are uncertain. However given the state x and time t_i, the set of possible jump sizes at this state is determined by the set $\mathcal{I}_k(x)$. The set-valued maps $\mathcal{I}_k$ may be given by the sub-differential of a lower semi-continuous convex functional ϕ_i. In this case, the system is governed by evolution inequations at the points of time t_k. Another situation that may give rise to such a dynamic model originates from the parametric uncertainty such as $\mathcal{I}_k(x) = \{I_k(t,x);\ t \in I\}$, where $\{I_k\}$ is a suitable family of functions $I \times E \to E$. To our knowledge, there are very few results for impulsive evolution inclusions with multi-valued jump operators; see [5, 19, 36]. The results of this book extend and complement those obtained in the absence of the impulse functions I_k, and for those with single-valued impulse functions I_k.

This book is arranged and organized as follows:

In Chap. 1, we introduce notations, definitions, and some preliminary notions. In *Sect. 1.1*, we give some notations from the theory of Banach spaces. *Section 1.2* is concerned to recall some basic definitions and some properties in Fréchet spaces. In *Sect. 1.3*, we recall some basic definitions and give some examples of Phase spaces. *Section 1.4* contains some properties of set-valued maps. In *Sect. 1.5*, we give some preliminaries about evolution systems. Some definitions and properties of the theory of semigroups are presented in *Sect. 1.6*. In *Sect. 1.6.3*, we give some properties of the extrapolation method. The last section (*Sect. 1.7*) contains some fixed point theorems.

In Chap. 2, we study some first order classes of partial functional, neutral functional, integro-differential, and neutral integro-differential evolution equations with finite delay on the positive real line. *Section 2.2* deals with the existence and uniqueness of mild solutions for some classes of partial evolution equations with local and nonlocal conditions. We give some results based on the fixed point theorem of Frigon in Fréchet spaces. An example will be presented at the last illustrating the abstract theory. In *Sect. 2.3*, we study some neutral differential evolution equations in Fréchet spaces. In *Sect. 2.4*, we give existence results for other classes of partial functional integro-differential evolution equations. *Section 2.5* deals with uniqueness results of neutral functional integro-differential evolution equations.

In Chap. 3, we provide sufficient conditions for the existence of the unique mild solution on the positive half-line $\mathbb{R}_+$ for some classes of first order partial functional and neutral functional differential evolution equations with infinite delay. In *Sect. 3.2*, we study the existence and uniqueness of mild solutions for partial functional evolution equations in Fréchet spaces. *Section 3.3* deals with the controllability of mild solutions on finite interval for partial evolution equations. In *Sect. 3.4*, we study the controllability of mild solutions on semi-infinite interval for partial evolution equations. *Section 3.5* deals with the existence of the unique mild solution of neutral functional evolution equations. In *Sect. 3.6*, we study the controllability of mild solutions on finite interval for neutral evolution equations. *Section 3.7* deals with the controllability of mild solutions on semi-infinite interval for neutral evolution equations.

In Chap. 4, we shall be concerned by perturbed partial functional and neutral functional evolution equations with finite and infinite delay on the semi-infinite interval $\mathbb{R}_+$. Our main tool is the nonlinear alternative proved by Avramescu (1.30) for the sum of contractions and completely continuous maps in Fréchet spaces [32], combined with semigroup theory. In *Sect. 4.2*, we study the existence of mild solutions for perturbed partial functional evolution equations with finite delay. *Section 4.3* deals with perturbed neutral functional evolution equations with finite delay. In *Sect. 4.4*, we study the existence of mild solutions for perturbed partial evolution equations with infinite delay.

In Chap. 5, we provide sufficient conditions for the existence of mild solutions on the semi-infinite interval $\mathbb{R}_+$ for some classes of first order partial functional and neutral functional differential evolution inclusions with finite delay. In *Sect. 5.2*, we study the existence of mild solutions for a class of functional partial evolution equations. *Section 5.3* deals with neutral partial evolution equations.

In Chap. 6, we study the existence of mild solutions on the semi-infinite interval $\mathbb{R}_+$ for some classes of first order partial functional and neutral functional differential evolution inclusions with infinite delay. In *Sect. 6.2*, we study functional partial evolution equations. *Section 6.3* deals with neutral partial evolution equations.

In Chap. 7, we are concerned by the existence of mild and extremal solutions of some first order classes of impulsive semi-linear functional differential inclusions with local and nonlocal conditions when the delay is finite in separable Banach spaces. Using a recent theorem due to Dhage combined with the semigroup theory, the existence of the mild and extremal mild solution are assured. The nonlocal case is studied too. In *Sect. 7.2*, we study the existence of mild solutions with local conditions. *Section 7.3* deals with the existence of mild solutions with nonlocal conditions. In *Sect. 7.4*, we give an application to the control theory.

In Chap. 8, we shall establish sufficient conditions for the existence of integral solutions and extremal integral solutions for some non-densely defined impulsive semi-linear functional differential inclusions in separable Banach spaces with local and nonlocal conditions. In *Sect. 8.2*, we give some results for integral solutions of non-densely defined functional differential inclusions with local conditions. *Section 8.3* deals with extremal integral solutions with local conditions, and *Sect. 8.4* deals with extremal integral solutions with nonlocal conditions. In *Sect. 8.5*, we give an application to the control theory.

In Chap. 9, we study the existence of mild solutions impulsive semi-linear functional differential equations. In *Sect. 9.2*, we study some existence results for semi-linear differential evolution equations with impulses and delay. *Section 9.3* is devoted to some classes of impulsive semi-linear functional differential equations with non-densely defined operators. In *Sect. 9.4*, we study impulsive semi-linear neutral functional differential equations with infinite delay. *Section 9.5* deals with integral solutions of non-densely defined impulsive semi-linear functional differential equations with state-dependent delay.

In Chap. 10, we shall establish sufficient conditions for the existence of mild, extremal mild, integral, and extremal integral solutions for some impulsive semi-linear neutral functional differential inclusions in separable Banach spaces. In *Sect. 10.2*, we study some densely defined impulsive functional differential inclusions. *Section 10.3* deals with the existence of mild solutions for non-densely defined impulsive neutral functional differential inclusions. In *Sect. 10.4*, we study the controllability of impulsive semi-linear differential inclusions in Fréchet spaces.

In Chap. 11, we study functional differential inclusions with multi-valued jumps. In *Sect. 11.2*, we study some existence of integral solutions for semi-linear functional differential inclusions with state-dependent delay and multi-valued jump. *Section 11.3* deals with impulsive evolution inclusions with infinite delay and multi-valued jumps. *Section 11.4* deals with impulsive semi-linear differential evolution inclusions with non-convex right-hand side. In *Sect. 11.5*, we study some impulsive evolution inclusions with state-dependent delay and multi-valued jumps. *Section 11.6* deals with the controllability of impulsive differential evolution inclusions with infinite delay.

In Chap. 12, we study functional differential equations and inclusions with delay. In *Sect. 12.2*, we prove some global existence for functional differential equations with state-dependent delay. *Section 12.3* deals with global existence results for neutral functional differential equations with state-dependent delay. In *Sect. 12.4*, we give some global existence results for functional differential inclusions with delay. *Section 12.4.1* deals with global existence results for functional differential inclusions with state-dependent delay.

In Chap. 13, we shall establish sufficient conditions for global existence results of second order functional differential equations with delay. In *Sect. 13.2*, we give some global existence results of second order functional differential equations with delay.

Keywords and Phrases: Evolution differential equations and inclusions, integro-differential equations, densely and non-densely defined differential equations, convex and non-convex valued multi-valued, mild solution, weak solution, initial value problem, nonlocal conditions, contraction, existence, uniqueness, measure of noncompactness, Banach space, Fréchet space, phase space, impulses, time delay, state-dependent delay, fixed point.

Contents