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The Quantum Case

Springer

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ISSN 0172-5939 ISSN 2191-6675 (electronic)
Universitext
ISBN 978-3-319-15828-0 ISBN 978-3-319-15829-7 (eBook)
DOI 10.1007/978-3-319-15829-7

Library of Congress Control Number: 2015935233

Mathematics Subject Classification (2010): 16W30, 17B37, 46L87, 81R60

Springer Cham Heidelberg New York Dordrecht London
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Printed on acid-free paper

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To Micho

Preface

First, who is the intended audience? I think I am addressing persons who either are active researchers in some area of mathematics or physics or are preparing themselves for that. Also, they are not experts in the topic of the title and want to learn something about it. As a nonexpert myself, I feel qualified in knowing what other nonexperts need to know. But absolutely everyone is welcome since science is open to all. However, experts in noncommutative geometry should find themselves rather bored. Among other things, I am trying to give a quite detailed mathematical presentation so that everything is accessible to physicists as well as mathematicians. It is easier for mathematicians to skip over details they do not need than for physicists to fill in seemingly enormous gaps that mathematicians barely notice even being there. And I do hope to find many physicists in my audience.

What are the prerequisites? Basically, gumption. With enough of that one can assimilate the knowledge base to get up to speed for page 1. What is that knowledge base? That is not so easy to specify since you might want to know what the *minimal* knowledge base is that's just enough to get by. Unfortunately, no one really knows what that might be. Of course, you have to know calculus, but somehow you absolutely need to know much more calculus than will actually be used here. Why? As a scientist, all I can say is I observe this to be the case. A true explanation[1] would seem to require an amazing amount of knowledge about how a human being acquires and maintains control over, well, an amazing amount of knowledge. But that explanation may well have something to do, for example, with how we go about understanding what a derivative is. The point is that learning the definition and perhaps the basic properties of derivatives is not going to do the trick for most folks. One has to continue to learn in greater depth before fully grasping the basics. Or as another saying goes, you only learn what course n is about by taking course $n + 1$.

Besides calculus, here is a short shopping list of prerequisites: linear algebra including tensor products, modern algebra (some keywords: rings, groups, free,

[1] I leave the definition of this as the first of many exercises for the reader.

homomorphisms, ideals), category theory up to functors and natural transformations (but as a language written in diagrams and not so much as a theory), and classical differential geometry. For the last topic the outstanding reference is [49] though other presentations are available, such as the companion volume [76] of this volume. Also, it is traditional to throw an escape clause into the deal: The reader should have mathematical maturity. Far be it from me to break from such well-established tradition.

The organization of the specific contents of this book and further relevant information are given in Chapter 1, the introductory chapter. But briefly, let me say that this book itself is just an introduction to quantum principal bundles.

The approach used here is mathematical in some sense that has gradually evolved over time from Cauchy's days, reaching a sort of climax—or anticlimax according to some—in the Bourbaki school. Personally, I am a great admirer of the 20th-century Russian school, which did not make such a big fuss about distinguishing mathematics and physics as other modern schools did. And I do have great respect for the accomplishments of Bourbaki & Co. in the 20th century. I only wish they had been even more ambitious.

As for acknowledgments, they extend way back. To give some context to this complicated question, let me explain a bit about where I come from. You see, I work in Mexico, a country in which I was not born. So once after beginning the very first day of a course here and explaining its general outline, I asked my students if there were any questions. And one student asked (*en español, por cierto*) where I come from. Of course, I meant questions about the course, not about me. So it was not an appropriate question. But after thinking about it a moment, I felt it was an appropriate question about my scientific background. So I answered: I come from Hilbert. It is not a complete answer, but I deserve partial credit since it is partially true. My acknowledgments go back even further, but clearly Hilbert is up there high on the list even though I never knew him. But some of my first teachers did know him or were indirectly influenced. I owe a great debt to them, especially J. Alperin, I. Kaplansky, A. Liulevicius, S. Mac Lane, and R. Welland, all of whom had a clarity of exposition that was thrilling and captivating for me. They set the standard for me early on. More recently, I have benefited greatly from my contacts with many colleagues but especially (again in alphabetical order) from J. Cruz Sampedro, L. Gross, B. Hall, L. Thomas, and C. Villegas-Blas, who have all reinforced the vision that mathematics is much, much more than formal manipulations of symbols—and that physics provides an illuminating beacon for identifying important mathematical research. These lists are not complete by any means. Think of them as my team's starting players. But I also had great strength on the bench, far too many to list. Thanks to all the unnamed heroes, too!

But as far as the project behind this particular book is concerned, my unending gratitude goes to Micho Đurđevich, who opened my eyes to the wonderful world of noncommutative geometry and quantum groups. For many years we tried to find common ground for our differing approaches to the infinite-dimensional world of mathematical physics, always with Micho's probing curiosity and marvelous patience. And eventually we did hit pay dirt. Professionally, Micho has been a

colleague and a collaborator *par excellence*. It is more than wonderful to count him as a great friend as well.

As for specific support for this book, I wish to thank Charles Dunkl for providing a copy of [20]. Also, the comments of the anonymous referees helped me to clarify my presentation of various points. The people at Springer, especially my editor, Donna Chernyk, get a lot of my thanks too.

As for errors, omissions, confusions, and other misdemeanors, I claim full innocence based on my present ignorance (= lack of knowledge) of any such offenses but plead guilty to them all and appeal for mercy. *¡Culpa mía! Nadie es perfecto*. My humble request is that all recriminations be limited to a message informing me of my trespasses, which are mine and mine alone. I meant no harm and hope no offense is taken.

Guanajuato, Mexico
September 2014

Stephen Bruce Sontz

Contents

Abbreviations

FODC	First-order differential calculus
HODC	Higher-order differential calculus
LIVF	Left invariant vector field
QCD	Quantum connection displacement
QED	Quantum electrodynamics
QPB	Quantum principal bundle
SR	Structure representation