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Stephen Bruce Sontz

Principal Bundles

The Classical Case

Springer

Stephen Bruce Sontz
Centro de Investigación en Matemáticas, A.C.
Guanajuato, Mexico

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A Lilia con muchísimo amor

Preface

Yet another text on differential geometry! But *why*? The answer is because this book is focused on one particular topic in differential geometry, that is, principal bundles. And the aim is to get the reader to an understanding of that topic as efficiently as possible without oversimplifying its foundations in differential geometry. So I aim for an early arrival at the applications in physics that give this topic so much of its flavor and vitality. But this is just the view from the classical perspective, which is why we speak of classical differential geometry (sometimes simply classical geometry) and refer to the topic of this volume as classical principal bundles.

There is a saying that all is prologue. As with any saying, it has a limited range of applicability. But here it is relevant since the topic of this volume serves as preparation for the corresponding quantum (or noncommutative) case, which will be the topic of the companion volume [44]. That is usually known as noncommutative geometry, and the corresponding bundles are known as quantum principal bundles. That is a newer field still being studied and refined by contemporary researchers. While none of this quantum theory has reached a stage of general consensus on "what it is all about" or on what is the "best" approach, we do know enough about the various approaches to be sure that certain common elements will dominate future research. And a lot of the intuition and motivation for the quantum theory comes from the classical theory presented in this volume, which serves as prologue.

Those are the goals of the two volumes. And as far as I am aware, these two texts together comprise the only published books devoted exclusively to the exposition of principal bundles in these two settings. That's *why*.

For *whom* are these books intended? The intended audience for these two volumes are folks who have some interest in learning topics of current interest in geometry and their relation with mathematical physics. I have basically two groups in mind.

The first group comprises mathematicians who have not seen the applications of principal bundles in physics. For them, the first part of this book should be more accessible since concepts are defined and theorems are proved, all according to the modern criteria of mathematical rigor. The second part of this book might require more effort for them since some amount of physical intuition is always helpful for understanding the applications.

The second group is physicists with some familiarity with the standard topics in physics (such as classical electrodynamics) but who have not seen how the mathematics used for those topics works in detail as an aspect of geometry. For this group, the first part of the book will likely require more work. So I have tried to include a lot of motivation and intuition to make that part more accessible for them. Consequently, my more mathematically inclined readers may at times find too many details for their level of expertise. Please be patient and understand why this is happening. In short, this book is meant to help bridge the communication gap between the two communities of physicists and mathematicians. In brief, that's for *whom*.

And *how*? I have tried to make the presentation throughout the book as intuitive as I possibly could. Ideas and intuitions are emphasized as being important elements in formulating the theory. I even think that intuition and ideas are more important than rigor, though they should not exclude rigor but rather precede it to give motivation. This is the "tricky bit," as the saying goes. But intuition is not a sufficient condition for getting things right, nor is rigor. Intuition in physics has led even Nobel laureates to arrive at ideas in contradiction to experiment. As an example, there were those who rejected the initial conjecture of parity being violated in the weak interactions. And mathematicians, even armed with rigor, have also fallen into error. The original "proof" of the four-color theorem comes to mind. Nobody is perfect!

Also, there are plenty of exercises to keep the reader active in the creation process that is essential to understanding, as Feynman so neatly puts it in the quote at the start of Chapter 1. The intention is that the exercises should give the reader hands-on experience with the ideas and intuitions. Some exercises are rather routine, while others are meant to challenge the reader. And I do not even tell you which ones are the routine exercises and which ones are the tough nuts to crack. All of that recognition is part of your own learning process. However, there is an appendix with further discussion of the exercises, including hints. I recommend that you hold off on looking at this (as well as the multitude of other texts on these topics) for as long as possible. If not longer. Even so, some problems may remain quite difficult. But this manner of presentation is intentional and is meant to help the reader in the long run.

I have been accused of asking the reader to write major parts of the book by doing difficult exercises with no hints or suggestions in the text itself. Well, that is true. The more of this book that you, my kind reader, can write, the better off you will be. And sometimes I even give an exercise before acquainting you with the usual tools required for solving it. So gratification is not always immediate. Welcome to the real world of science! This is meant to be a difficult book, much more so than the usual introductory technical texts on the market. That's *how*.

I have avoided a strictly historical approach since the way we have arrived at this theory can obscure its logical structure. For example, the Dirac monopole appeared in the 1930s but is presented here near the end of the book as a special case of the theory developed decades later.

Mathematicians should be aware that physics is a discipline with its own logical structure, though that is not always clear to mathematicians. Ideas flow into other ideas. Results that involve very particular ideas (such as the Yang–Mills theory) then motivate generalizations, which in turn illuminate the original particular case (among other examples) in new ways. This is not the paradigm of definition, theorem, proof. Nonetheless, it is a logical structure in its own right. In the sections on applications to physics, I have tried to include a lot more than the usual amount of motivation based on physics ideas in the hope of helping my mathematically inclined readers. The physicists among my readers might find this boring, and so I kindly request their patience and understanding. But they might be more challenged by the translation of these ideas into a mathematically rigorous geometric formulation.

These are texts meant for learning the material, for either actual students or others who want to learn about principal bundles. The pace is designed for use in a course or for self-study. My experience is that there is enough material here for a one-year graduate course although undergraduates with an adequate background and motivation could profit from such a course.

These are not definitive treatises meant only for the purpose of giving experts a place to look up all the variants of theorems. The experts should not expect too much from either of these volumes, except perhaps as a way of organizing these topics for their own courses. I have not included a multitude of fascinating topics, both in geometry and in physics. These introductory texts should serve to motivate the reader to carry on with study and research in what is *not* presented as well as in what is presented.

The prerequisites for this volume consist of a bit of many things, such as the basic vocabulary of group theory (not to be confused with group therapy); a smattering of linear algebra, including tensor products (even though this will be briefly reviewed); multivariable calculus of real variables, including vector calculus notation; at least a vague appreciation of what a nonlinear differential equation is; some talking points from elementary topology (such as compact, open, closed, Hausdorff, continuous, etc.); and something about categories as a system of notation and diagrams, but not as a theory. And a bit of physics for the chapters with such applications may be useful though I have tried to keep that material as self-contained as possible.

More than anything else, the present volume is my own personal take on classical differential geometry, the role that principal bundles play in it, and how all this relates to physics. Since the goal of the book, as its title reveals, is the exposition of the theory of principal bundles in the classical case, not all of the quite fascinating topics of differential geometry will be presented in the first chapters, but more than enough to get us to that goal. But to understand what principal bundles are "good for" requires meaningful examples as well. And since my personal motivation comes from physics, I devote the remainder of the book to several chapters on examples taken from physics. These chapters make this book more than just another introduction to principal bundles. I tried to make these as self-contained in terms of the physics content as I could, but I rely heavily on the mathematical theory developed earlier in the book. I have always felt that learning physics with little more than a solid mathematics background is far easier than the other way around. So I beseech my physicist readers not to despair during the first part of this book. The trip may be tougher than you'd like, but the payoff should be well worth it.

Nothing here is original. I can merely hope that my way of presenting this material has an appealing style that makes things intuitive and accessible. This book is my own personal response to the challenge Richard Feynman posed to himself, and to no one else, in the famous last blackboard quote cited in Chapter 1.

This is also the place to give thanks to all those who helped me create my own understanding of these topics. It turns out to be the sum of many contributions, some small, some quite large. And all over a long period of time. Many are people whom I have never met but only know through their publications. Names like Courant and Robbins come to mind since they were the first to show me how to think about mathematics in their book [5]. Others have spoken directly to me, but about topics that seem to be worlds away from differential geometry. A name like Larry Thomas comes to mind since he was the first to show me how to think about mathematical research. Also, when I was a graduate student at Virginia, David Brydges gave a very pretty course about gauge theory that helped me put a lot of details that I vaguely knew into sharper focus. The list goes on and on. Please, my friends and colleagues, do not be offended that you are not explicitly mentioned here. The list is way too long.

Also, I gratefully thank all those at Springer who produced a book out of a manuscript. Among those, special thanks go to Donna Chernyk, my editor. The comments of the anonymous referees also helped me improve the book. For those not aware of the details of this conversion process, let me say that it is neither a function nor a functor as far as I can figure out, but it is certainly a lot of work.

This volume is based on several introductory graduate courses that I have taught over the years. Among the participants in those courses, very special thanks are due to Claudio Pita for his helpful comments and continual insistence on ever-clearer explanations.

But there is one person who ignited the spark that made differential geometry quite clear from the very start. And that is Arunas Liulevicius. I most graciously thank him. He taught the first courses I ever took on differential geometry. His intelligent, rigorous discourse, blended with an authentic, bemused humor, made a great impact on me as a model of how to approach mathematics in general, not just these topics. And I never had to un-learn anything—which is a sort of litmus test in itself too! In terms of the specific topics in the beginning of this book, I am more indebted to him than to anyone else.

As for errors, omissions, and all other sorts of academic misdemeanors, there is no one to blame but myself. Please believe me that though I be guilty of whatever shortcoming, I am innocent of any malicious intent. In particular, omissions in the references are mere reflections of my limited knowledge. I request my kind reader to help me out with a message to set me right. I will be most appreciative.

Guanajuato, Mexico — Stephen Bruce Sontz
September 2014

Contents

Abbreviations

ASD	anti-self-dual
BPST	Belavin, Polyakov, Schwartz, Tyupkin
EL	Euler–Lagrange
CR	Cauchy–Riemann
GR	general relativity
LIVF	left invariant vector field
ODE	ordinary differential equation
QCD	quantum chromodynamics
SD	self-dual
SI	Système International d'Unités

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