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Formal Algorithmic Elimination for PDEs

 Springer

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Preface

This monograph originates in my habilitation treatise. The theme is to understand the correspondence between systems of partial differential equations and their analytic solutions from a formal viewpoint. We consider on the one hand the problem of determining the set of analytic solutions to such a system and on the other hand that of finding differential equations whose set of solutions coincides with a given set of analytic functions. Insight into one of these problems may advance the understanding of the other one. In particular, as shown in this monograph, symbolic solving of partial differential equations profits from the study of the implicitization problem for certain parametrized sets of analytic functions.

Two algorithms are fundamental for this work. Janet's and Thomas' algorithms transform a given system of linear or nonlinear partial differential equations into an equivalent form allowing to determine all formal power series solutions in a straightforward way. Moreover, these effective differential algebra techniques solve certain elimination problems when applied in an appropriate way. The first part of this monograph discusses these methods and related constructions in detail. Efficient implementations of both algorithms in the computer algebra system Maple have been developed by the author of this monograph and his colleagues.

The second part of this monograph addresses the problem of finding an implicit description in terms of differential equations of a set of analytic functions which is given by a parametrization of a certain kind. Effective methods of different generality are developed to solve the differential elimination problems that arise in this context. As a prominent application of these results it is demonstrated how some family of exact solutions of the Navier-Stokes equations can be computed.

I would like to thank several persons for many fruitful discussions about algebra, about PDEs, and their connection, and for supporting my work. In particular, I would like to express my gratitude to Prof. Dr. W. Plesken, Prof. Dr. J. Bemelmans, Prof. Dr. C. Melcher, Prof. Dr. S. Walcher at RWTH Aachen and Priv.-Doz. Dr. M. Barakat at TU Kaiserslautern. Thanks go to Prof. Dr. V. P. Gerdt (JINR Dubna) and Dr. A. Quadrat (Inria Saclay), not only because I have been enjoying our collaborations, but also for reading parts of a first version of this monograph.

In the context of work on Thomas' algorithm, I am grateful to my colleagues Dr. T. Bächler and Dr. M. Lange-Hegermann for their help. Moreover, we profited very much from discussions with Prof. Dr. D. Wang (Paris 6), Prof. Dr. F. Boulier, and Dr. F. Lemaire (both at Université de Lille) on this topic.

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Daniel Robertz

Contents

1	Introduction	1
2	Formal Methods for PDE Systems	5
2.1	Janet's Algorithm	5
2.1.1	Combinatorics of Janet Division	9
2.1.2	Ore Algebras	16
2.1.3	Janet Bases for Ore Algebras	21
2.1.4	Comparison and Complexity	37
2.1.5	The Generalized Hilbert Series	40
2.1.6	Implementations	54
2.2	Thomas Decomposition of Differential Systems	57
2.2.1	Simple Algebraic Systems	60
2.2.2	Simple Differential Systems	88
2.2.3	The Generic Simple System for a Prime Ideal	104
2.2.4	Comparison and Complexity	110
2.2.5	Hilbert Series for Simple Differential Systems	112
2.2.6	Implementations	116
3	Differential Elimination for Analytic Functions	119
3.1	Elimination	119
3.1.1	Elimination of Variables using Janet Bases	121
3.1.2	Complexity Reduction by means of a Veronese Embedding .	128
3.1.3	Elimination of Unknown Functions using Janet Bases	134
3.1.4	Elimination via Thomas Decomposition	139
3.1.5	Compatibility Conditions for Linear Systems	147
3.2	Linear Differential Elimination	160
3.2.1	Implicitization Problem for Linear Parametrizations	161
3.2.2	Differential Elimination using Linear Algebra	165
3.2.3	Constructing a Representation	178
3.2.4	Annihilators for Linear Parametrizations	182
3.2.5	Applications	186

- 3.3 Nonlinear Differential Elimination 192
 - 3.3.1 Implicitization Problem for Multilinear Parametrizations . . . 194
 - 3.3.2 Elimination Methods for Multilinear Parametrizations 198
 - 3.3.3 Implicitization using Kähler Differentials 207
 - 3.3.4 Constructing a Representation 216
 - 3.3.5 Applications 221

- A Basic Principles and Supplementary Material 233**
 - A.1 Module Theory and Homological Algebra 233
 - A.1.1 Free Modules 233
 - A.1.2 Projective Modules and Injective Modules 235
 - A.1.3 Syzygies and Resolutions 238
 - A.2 The Chain Rule of Differentiation 243
 - A.3 Differential Algebra 247
 - A.3.1 Basic Notions of Differential Algebra 247
 - A.3.2 Characteristic Sets 249
 - A.3.3 Homogeneous Differential Polynomials 253
 - A.3.4 Basis Theorem and Nullstellensatz 256

- References 259**

- List of Algorithms 273**

- List of Examples 275**

- Index of Notation 277**

- Index 279**