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Dynamics and Control of Trajectory Tubes

Theory and Computation

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Preface

This book presents a systematic treatment of theoretical methods and computation schemes for solving problems in dynamics and control. The mathematical models investigated here are motivated by processes that emerge in many applied areas. The book is aimed at graduate students, researchers, and practitioners in control theory with applications and computational realizations.

The emphasis is on issues of reachability, feedback control synthesis under complex state constraints, hard or double bounds on controls, and performance in finite continuous time. Also given is a concise description of problems in guaranteed state estimation, output feedback control, and hybrid dynamics using methods of this book. Although its focus is on systems with linear structure, the text also indicates how the suggested approaches apply to systems with nonlinearity and nonconvexity.

Of primary concern is the problem of *system reachability*, complemented by the problem of *solvability*, within a given class of controls. This leads to two basic questions: find the terminal states that the system can reach from a given initial state within a specified time, using all possible controls, and find the initial states from which the system can reach a given terminal ("target") set in specified time, using all possible controls. The answer to these questions takes the form of bundles of controlled trajectories—the reachability tubes—that emanate either forward in time from a given starting set (*the forward reachability tube*) or backward in time from a given target set (the solvability or *backward reachability tube*). The cross-sections ("cuts") of these tubes are the reachability ("reach") sets. Backward reach sets are important in designing *feedback (closed-loop) controls*.

We are thus led to dealing with *trajectory tubes*—set-valued functions that are the main elements of the required solutions. These tubes are also key in describing systems operating under uncertainty, in the form of unknown but bounded parameters in the system model, inputs, and measurement errors. The mathematical properties of the solutions, such as non-differentiability, may also lead to set-valued functions. Hence we have to conduct the investigation of feedback control system dynamics taking into account the set-valued nature of the functions involved. This is achieved using the Hamilton–Jacobi formalism in conjunction with the Dynamic Programming approach and studying value functions for appropriate problems of dynamic optimization. These value functions turn out to be the solutions to related types of the HJB (Hamilton–Jacobi–Bellman) equation with appropriate boundary conditions. The level sets of such value functions, evolving in time, describe the tubes we need.

In addition to theoretical rigor, the solutions to modern control problems must be effectively computable. That is, the results must be available within prescribed time, perhaps on-line, the set-valued trajectory must be calculated with a guaranteed accuracy, and the computation procedures should be able to handle high-dimensional systems. This may require parallelization of solutions and distributed computation. The computation approaches presented here meet such needs. They are based on ellipsoidal calculus introduced in [174], successfully extended and applied in [181, 182], and complemented by appropriate software tools [132].

The book is divided into eleven chapters. Chapter 1 provides an exposition of problems in control theory for linear systems with solutions given in the specific forms needed for transition to computable operations over trajectory tubes.

Chapter 2 is focused on *how to solve*. The main theoretical tools—Hamiltonian methods and Dynamic Programming techniques—as applied to problems of reachability and target control synthesis are described. These are reduced to optimization problems, whose solution is given by an equation of the HJB type. In its turn, the solution to the HJB equation produces value functions whose level sets are the desired reach sets. Usually, attention is needed in interpreting the solution to the HJB equation. However, for "linear-convex" systems (with convex constraints on the controls and convex starting or terminal sets) the value functions are convex in the state variables and usually unique. These functions are directionally differentiable along any direction in the state space. Hence there is no need for subtle definitions of generalized viscosity solutions since for equations used here they may be expressed through conventional classical arguments. Moreover, direct integration of the HJB equation is avoided by calculating the *exact* value function through duality methods of convex analysis. Such a procedure is also used to apply a verification theorem to the concrete HJB equation, confirming that the calculated value function is its solution. This also allows us to derive the feedback control directly from the HJB equation. The derived control is in general a set-valued function and its substitution in the system equation produces a *differential inclusion*. Effective computation of trajectories that steer the system from point to point arrives through an intersection of colliding tubes—the forward reach tube from a given starting point and the backward reach tube from a given terminal point. The final section describes a solution to the problem of *time-optimal* target control.

Chapter 3 indicates *how to calculate* the solutions given as set-valued functions. The approach develops external and internal ellipsoidal-valued functions that approximate the reachability tubes. Parametrized families of ellipsoids are designed, whose intersection produces an external bound and whose union produces an internal bound. Increasing the number of approximating ellipsoids in the limit yields the exact reach set. The next move is to calculate the synthesizing feedback target control. As already mentioned, such a feedback strategy may be calculated directly from the HJB equation. However, this control strategy, being a nonlinear function of time and state, would require an appropriate existence theorem for the synthesized nonlinear differential inclusion. To avoid this possible difficulty, it is convenient to use the "aiming" rule of Krasovski. Namely, calculating the backward reachability set in advance, one should design the feedback control such that it keeps the trajectory within the backward reach tube, following this "bridge" until it reaches the terminal target set. The computation scheme for the aiming rule is as follows: with given starting point in "time-space" that lies within or beyond the backward reach set, one selects an internal ellipsoidal approximation of the backward set that either contains this point or is the closest to it. This aiming strategy may be calculated explicitly or with minimal computational burden.

Chapter 4 offers examples of problems solved by ellipsoidal methods and illustrated graphically. Though attractive and effective, ellipsoidal solutions may run into some degenerate cases that are more common for internal approximations. These potential computational problems are especially evident in large systems where the dimension of controls is much smaller than the system dimension. Ways of *regularizing* such situations are discussed in a section on high-dimensional systems. The described tools have proved effective in computational experiments. The treatment of high-dimensional systems indicated here is reached through a parallelization of solutions, which is a natural extension of the suggested schemes.

The first part of Chap. 5 is devoted to *nonlinearity* and *nonconvexity*. The general approach formulated here is applicable to nonlinear systems and presented in the form of a *comparison principle*. The idea is to approximate the available HJB equation from above or below through relations that ensure guaranteed upper or lower solution estimates by operating with functions simpler than in the original equation. This approach does *not* depend on the type of exact solution to the HJB equation, whether classic or generalized. At the same time, the comparison principle allows one to develop a *deductive approach* to the ellipsoidal calculus of this book in contrast with the *inductive approach* of Chap. 3. These topics are followed by examples of reachability for nonlinear systems and calculation of the set of points *reachable within a time interval*, which is a nonconvex union of convex sets. The second part of the chapter deals with the application of ellipsoidal methods to systems with *non-ellipsoidal constraints*, such as boxes and zonotopes (symmetrical polyhedra).

The chapters that follow cover a variety of useful properties and specific problems related to implementing the proposed approach. Chapter 6 emphasizes the role of *double constraints*—the simultaneous presence of both a hard bound and an integral bound on the controls. Solutions to such problems are needed for treating *impulse controls* investigated in this chapter. Impulse inputs were previously studied as open-loop controls with few special results on feedback solutions [23,36]. A theory of feedback impulse control is presented here for systems of arbitrary dimension, with solutions in terms of *quasi-variational inequalities* of the HJB type. However, impulse inputs are ideal elements and their physical realization by "ordinary" bounded functions may be achieved through approximation by double-bounded controls which are functions whose bound tends to infinity.

Chapter 7 is concerned with dynamics and control of systems under *state constraints*. Previously discussed problems are now subject to additional "viability" restrictions on the state variables. The description covers solution approaches to problems of reachability (forward and backward) and emphasizes specifics of related mathematical techniques including ellipsoidal approximations. The case of linear systems with convex hard bounds on controls and state coordinates are worked out in detail.

The contents of Chaps. 1–7 indicate that one of the main items in treating considered problems are trajectory tubes and the means of their calculation. The same is true for the rest of the chapters. So Chap. 8 begins with fundamentals of a general vision—*the theory of trajectory tubes* with models of their evolutionary dynamics. Indicated results, together with considerations of the previous chapter, are further applied to *closed-loop control under state constraints*, with techniques borrowed from both Hamiltonian approach and duality theory of nonlinear analysis. This brings forward the discussion to complex state constraints in the form of *obstacle problems*, wherein constrained trajectories must simultaneously lie *within* one set and *outside* another.

The next two chapters consider *uncertainty*, which is inherent in realistic problems of control. These chapters may serve as an introduction to a more thorough description of uncertainty.

Chapter 9 is a concise explanation of the *theory of guaranteed state estimation*, also known as the *set-membership bounding approach* to external disturbances in estimation models. In contrast with conventional descriptions, the present exposition involves Hamiltonian methods and is applicable to nonlinear systems. Dynamic estimation of system trajectories under unknown but bounded errors is also formulated as a problem with state constraints, which now are not known in advance, but arrive online, in real time. In the linear case the proposed deterministic "filtering" equations demonstrate connections and differences when compared with stochastic (Kalman) filtering. Both continuous and discrete measurements are considered.

The results of Chap. 9 have a natural application to problems of *output feedback control* under unknown but bounded disturbances in system and measurement inputs. These problems are addressed in Chap. 10. The solutions introduced there are based on the notions of *generalized state* and *information tubes* which describe the overall system dynamics. For the linear case with convex constraints on controls and uncertainties items, computation schemes based on ellipsoidal approximations are presented. Several examples are worked out.

Finally, Chap. 11 is confined to *verification* problems and *hybrid systems*, with a description of exact solutions and ellipsoidal schemes for their computation. The discussion is accompanied by some examples. Special attention is given to possible involvement of impulse inputs in formal mathematical models of hybrid systems. The aim of this chapter is to emphasize the applicability of techniques presented in this book to the investigation of hybrid systems.

References to prior literature and research results related to this book are given in the introduction to each chapter.

The mathematical level of the book presumes reasonable knowledge of advanced calculus, linear algebra and differential equations, as well as basics of variational analysis with optimization methods and computational mathematics.

Throughout the previous years we had the pleasant opportunity for useful discussions on the topics arranged in this book with K. Åström, J.-P. Aubin, J. Baras, T. Başar, F. Chernousko, P. Kokotovic, A. Krener, A.A. Kurzhanskiy, Yu. Ledyayev, G. Leitmann, A. Lindquist, J. Lygeros, M. Milanese, I. Mitchell, S. Mitter, A. Rantzer, J. Sousa, C. Tomlin, I. Vályi, and V. Veliov. Their valuable comments helped to crystallize the contents.

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Contents

Notations

$\mathbb{R}^n$—the n-dimensional Euclidean vector space, $\mathbb{R}^1 = \mathbb{R}$

$\mathbb{R}^{n\times m}$— the linear space of $n \times m$-matrices

$\langle x, y\rangle = x'y$—the scalar (inner) product of vectors $x, y \in \mathbb{R}^n$, with prime as the transpose

$\|x\|_M = \langle x, My\rangle^{1/2}, \quad M = M' > 0$

$\|x\| = \|x\|_I$, I—the identity (unit) matrix

$\mathrm{co}\,Q, \mathrm{conv}\,Q$—the convex hull of set Q

$\mathrm{comp}\mathbb{R}^n$—the variety of all compact subsets $Q \subset \mathbb{R}^n$

$\mathrm{conv}\mathbb{R}^n$—the variety of all convex compact subsets $Q \subset \mathbb{R}^n$

$d(x, Q) = \inf\{\|x - y\| \mid y \in Q.\}$—the distance of point x from set Q

$h(P, Q) = \max\{h^+(P, Q), h^+(Q, P).\}$—the Hausdorff distance between sets P, Q

$h^+(P, Q) = \max\{d(x, Q) \mid x \in P\}$—the Hausdorff semi-distance

$G(t, s)$—the fundamental transition matrix for a linear homogeneous system $\dot{x} = Ax$

int X—interior of set X

$I(l|B)$—indicator function of set B: $I(l|B) = 0$ if $l \in B$, $I(l|B) = \infty$ if $x \bar{\in} B$

$\rho(l \mid Q) = \sup\{\langle l, x\rangle \mid x \in Q\}$—the support function of set Q at point $l \in \mathbb{R}^n$

graph $Y(t) = \{\{t, y\} \in T \times \mathbb{R}^n \mid t \in T = [t, \vartheta], y \in Y(t)\}$

epi $f(x) = \{(x, y) \in \mathbb{R}^n \times \mathbb{R} \mid y > f(x)\}$—epigraph of function f

Dom $f = \{x : f(x) < \infty, \}$—effective domain of f

co$\varphi(l)$—closed convex hull of φ

$\partial\varphi(l)$—subdifferential of convex function $\varphi : \mathbb{R}^n \to \mathbb{R}$ at point l

$f^*(l) = \sup_x\{\langle l, x\rangle - f(x)\}$—Fenchel conjugate of function $f(x)$

V_x—partial derivative in x of function V

$C^m[T]$—the space of m-dimensional continuous vector functions $f : T \to \mathbb{R}^m$

$\mathcal{L}_p^m[T]$—the space of m-dimensional vector functions $f : T \to \mathbb{R}^m$ integrable with power p

$\mathbf{V}^m[T]$—space of m-dimensional vector functions of bounded variation on the interval $T = [t_0, \vartheta]$

$VarU(t), t \in T$—total variation of function $U \in \mathbf{V}^m[T]$, over the interval $t \in T$

$\|f(\cdot)\|_{\mathcal{L}_p}$—the norm of function $f(\cdot)$ in infinite dimensional space $\mathcal{L}[T]$