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Probability on Compact Lie Groups

Foreword by Herbert Heyer

Springer

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This book is dedicated to my family

Symmetry is a vast subject, significant in art and nature. Mathematics lies at its root, and it would be hard to find a better one on which to demonstrate the working of the mathematical intellect

Hermann Weyl, "Symmetry"

You should meditate often on the connection of all things in the universe and their relationship to each other

Marcus Aurelius, "Meditations"

Foreword

Since the trendsetting monographs of Grenander, Parthasarathy and Heyer the study of probability measures on algebraic-topological structures such as topological semigroups, groups and hypergroups has developed into a comprehensive area of mathematical research, with applications ranging from the dynamics of stochastic processes to statistical decision theory. A basic tool in advancing the theory was the abstract harmonic analysis of locally compact groups, in particular the technique of Fourier transformation based on unitary representations. While the initial studies mostly dealt with general classes of groups and measures, the emphasis during the last four decades has been concretization in two directions: to choose special classes of groups and to consider special types of measures. In the seminal books of Diaconis, Hazod and Siebert, and Liao innovative contributions about random walks on, and Lévy processes in groups were included up to the actual state of the art. Originally, there was a broad variety of problems of classical probability theory that called for generalization and standardization within the framework of locally compact groups. We mention selectively the embedding of infinitely divisible measure into convolution semigroups, the canonical decomposition of convolution semigroups, the central limit problem and arithmetical characterizations of prominent types of probability measures. All of these problems could be more easily approached once the underlying group was Abelian, although some of the analysis had already been extended beyond this special case at an early stage. The author of the present book chooses compact Lie groups in order to describe recent advances of the theory. On the basis of current sources he discusses for example results on positive definiteness of mappings on duals of compact Lie groups, on regularity of densities of probability measures, and on deconvolution. Part survey and part treatise, the presentation supplements the existing expository literature on the subject. It invites the reader to learn first about compact Lie groups and their representations before it leads him to topics in structurally oriented probability theory. A wealth of references accompanying the exposition opens the eye of the reader towards future research. It is a pleasant experience to see well-established results and recent progress in the theory of probability measures on compact Lie groups summarized in book form. Of course in a handy publication the choice of relevant topics can only be subjective. Yet the author's very own research and his overall picture of the field guarantee successful reading for all who wish to absorb research in probability theory from an algebraic-topological point of view.

Herbert Heyer

Acknowledgments

Most of this book was written in the academic year 2011–2012 when the author was granted study leave by the University of Sheffield. He would like to thank Tony Dooley and the University of New South Wales for hospitality in January–February 2012 where much progress on this project was made despite the attractions of Coogee Beach and cricket on the TV nearly every night! He would also like to thank Marta Sanz-Solé and the University of Barcelona for supporting a ten-day visit in May 2012, where he continued to make progress despite the rather different attractions of Catalan culture and tapas. Thanks are also due to many colleagues for discussions on topics relating to material in the book including Rodrigo Bañuelos, Fabrice Baudoin, Tony Dooley, Neil Dummigan, Herbert Heyer, Peter Kim, Fionntan Roukema and René Schilling. My profound thanks go to Michael Bingham, who read through the document with great care and made many extremely valuable (and sometimes rather deep) comments. The customary free copy of the published volume is a small reward for his labours. Thanks are also due to Ming Liao, Jonathan Manton and Markus Riedle, who each read and made helpful comments on particular chapters, to Christian Fonseca Mora, who very thoroughly worked through the whole draft manuscript and helped me remove many typos, and to Nick Bingham who provided me with a "full red-ink job", that enabled me to correct a host of grammatical inaccuracies. I would also like to thank two anonymous referees who gave me extremely valuable feedback that has led to many improvements. Thanks are also due a second time to Herbert Heyer for agreeing to write the Preface to this volume. Finally, it is a pleasure to thank my editor Marina Reizakis, and all the staff at Springer who contributed towards the production of the book.

Contents

Introduction

Probability on groups gives a mathematical context to the interplay of *chance* with *symmetry*. To study this subject involves investigating probability measures on groups and random variables, stochastic processes and random fields that are group-valued. At the very least this entails the interaction of (probabilistic) measure theory and group theory, but the range of mathematical tools required to develop the subject is much wider than this and includes representation theory, harmonic analysis, functional analysis and differential geometry, as well as stochastic analysis and stochastic differential geometry. Furthermore the subject is highly applicable, and in the first two decades of the twenty-first century there has been considerable interest in applications to engineering, such as signal processing, robotics and the measurement of biological molecules. There have also been complementary developments in statistical estimation and inference on groups and manifolds which have been strongly motivated by these applications.

We can study probability theory in, e.g. finite groups, algebraic groups, Lie groups, locally compact groups, loop groups and current groups, so the reader may well be asking—why *compact* Lie groups? There are a number of reasons for this. To begin with, this book is designed to be an *introduction*, and so it should avoid the temptations of generality. Indeed, the topic of probability theory on general locally compact groups is the theme of a classic monograph by Herbert Heyer [95], which although published in 1977 still remains highly relevant today. Compact Lie groups have the advantage that:

- They are the simplest class of continuous groups, beyond the abelian, to display the key feature of *noncommutativity* (in general) that leads to so much fascinating structure.
- They have a rather straightforward representation theory. In particular, their irreducible representations are all finite-dimensional. This leads to a *Fourier transform* (characteristic function) that is matrix-valued, while for more general groups this will operate in an infinite-dimensional space. Furthermore, the theory of highest weights enables us to carry out standard analytic operations (such as taking limits) in the "Fourier co-variable".
- Every sequence (or net) of probability measures on a compact group is tight.

- There are many important and interesting examples. These include the tori Π^n, the special orthogonal groups $SO(n)$ and the special unitary groups $SU(n)$. The group $SO(3)$ of rotations in three-dimensional space is of particular interest to engineers [48, 46, 47] and cosmologists [147]. The symmetry group of the standard model in elementary particle physics is the compact Lie group $U(1) \times SU(2) \times SU(3)$ [16].

Of course there is much that can be done with compact groups alone, but compact Lie groups admit a differential structure, and this enables us to exploit natural analytic/geometric structures such as the Laplacian and its associated heat semigroup and heat kernel. In fact, one can think of a Lie group informally as the natural context in which calculus meets symmetry, so one can, for example, describe symmetry transformations that preserve some invariant quantity, infinitesimally.

There are three key tools that are needed to study a probability measure μ or a process on a group G. The first is the Fourier transform as mentioned above, whose value at an irreducible representation π is $\widehat{\mu}(\pi) = \int_G \pi(g^{-1})\mu(dg)$. The second is functional analysis, which enters the picture when we consider convolution operators of the form $P_\mu f = f * \mu$ acting on various Banach spaces. Finally, we use stochastic analysis to study the interaction of systems with noise by means of stochastic differential, or stochastic partial differential equations. In this book we will almost entirely concentrate on the interplay between the first two of these tools and we will hardly use the third at all. Again there are some good reasons for this. First, this is an introductory book and I didn't want it to get too long. Secondly, from a pedagogic viewpoint, there is something to be said for imitating the process we go through as an undergraduate or first-year graduate student, where we take a course that covers topics like characteristic functions and the central limit theorem before we try to learn about stochastic integration. Thirdly, many topics that interest engineers and statisticians, such as deconvolution, seem to rely mainly on Fourier transform techniques. Finally, there was some intellectual curiosity on my own part—how far could I go without using one of my favourite tool kits? But the reader who gets to the end of the book will have encountered many places where stochastic analysis is needed to make further progress, and there are abundant references for further reading in this direction.

This book is specifically aimed at graduate students who have taken a standard course in measure-theoretic probability and have some training in functional analysis (particularly, Hilbert spaces) and a smattering of general topology, but do not necessarily have a strong background in group theory, representations and differential geometry. Of course it is also suitable for experts in Lie theory and associated harmonic analysis who want to find out what the probabilists are doing with their beautiful subject. Finally, I hope it will be accessible to the growing number of highly mathematically trained engineers, physicists and statisticians who are now working on applications of these ideas and who wish to fill in some gaps in their knowledge.

Guide to Reading the Book

Chapter 1 is a roller-coaster ride through the essentials of topological groups and Lie groups. Of course this is a huge topic and it cannot be learned in a hurry. I would encourage readers who are novices in this area to study an authoritative text as an accompaniment. Chapter 2 develops all the representation theory that we need for the book and proves one of the key results—the Peter–Weyl theorem. We also begin the important study of the Fourier transform of functions. Here I am highly indebted to Faraut's beautiful monograph [63], and large parts of the account are closely based on his approach. In the last part of the chapter, I introduce (in a somewhat condensed manner) the theory of weights and sketch the proofs of the wonderful character and dimension formulae of Hermann Weyl. Experts on Lie theory can skip these two chapters, but might want to glance at Sect. 2.3 to remind themselves of key properties of the Fourier transform.

The short Chap. 3 introduces the important tool of the Laplacian and the associated Sobolev spaces. In the second part of the chapter we utilise the one-to-one correspondence between irreducible representations and dominant weights to regard the Fourier transform as a "function" of the latter. Since weights live in a finite-dimensional vector space, we can now carry out standard analytic operations, such as taking limits, on the "Fourier co-variable". In particular, we present Sugiura's [200] far-reaching characterisation of smooth functions on the group in terms of decay properties at infinity of the Fourier transform.

In Chap. 4, we at last turn our attention to probability measures on groups and make a detailed study of characteristic functions (i.e. Fourier transforms of measures). In particular, we describe a compact-group analogue of the famous Lévy convergence theorem that is due to Kawada and Itô [114]. A key theme of this chapter is absolute continuity, and we begin to see the importance of convolution operators through a beautiful theorem of Raikov [164] that tells us that the probability measure μ is absolutely continuous if and only if the associated convolution operator P_μ is compact in the space of continuous functions on the group. Using Sugiura's techniques, we then find necessary and sufficient conditions for a measure to have a smooth density in terms of the decay of the characteristic function. In the last part of the chapter we study random walks on general (not necessarily compact) Lie groups, and give some potential-theoretic characterisations of recurrence due to Guivarc'h et al. [75].

Standard Gaussian measures on compact Lie groups are naturally described through their Fourier transform in terms of eigenvalues of the Laplacian (the Casimir spectrum). They have smooth densities which can be easily obtained from the heat kernel. In recent years there has been increasing interest in both theoretical work and modelling with non-Gaussian processes, and Lévy processes (which are essentially stochastic processes with stationary and independent increments) have been at the heart of this activity. In Chap. 5, we study the analogues of these processes on general Lie groups but at the level of measures. So we investigate convolution semigroups of probability measures and the associated C_0-semigroups of convolution operators. We obtain Hunt's important characterisation [103] of the generators of these semigroups, which can be seen as an analogue of the classical Lévy–Khintchine formula on Euclidean space. As a corollary we derive a less well-known and more direct generalisation of the Lévy–Khintchine formula using the Fourier transform. We also develop Breuillard's [35] presentation of the central limit theorem, itself originally due to Wehn. Subordination is an important technique for obtaining new examples of interesting convolution semigroups, and specialists in mathematical finance know that the most interesting cases are obtained by subordinating Brownian motion. The same is true for Lie groups, and we examine some important classes (such as analogues of the stable laws) and establish smoothness of densities.

Finally Chap. 6 is a short introduction to statistics on compact Lie groups. It focusses on work by Kim and Richards [118] on deconvolution in the context of estimating the distribution of a signal from observations made on an output that is distorted by an independent noise.

Each chapter finishes with a brief guide to further reading, and there are many references in the bibliography for readers to delve further into the literature. I should perhaps apologise to anyone whose work I have neglected to mention. The bibliography, like the book that it references, is not meant to be an exhaustive archive of contributions to this very wide area. It is rather a guide to the available literature, and in many cases I have referenced particular papers and books in the knowledge that the reader can find many more interesting references in the bibliography of that paper or book. Despite my best efforts, this book will surely contain many typos and (hopefully minor) errors. Please send these to me at d.applebaum@sheffield,ac.uk. They will be posted on my website at http://www.applebaum.staff.shef.ac.uk/books.html.

The book contains many proofs but in addition a large number of statements are included without proof, especially in Chap. 1. In particular I have tended to omit proofs that require a lot of structure theory, but include those that are more analytical, since I believe that this will make the volume more accessible for the typical reader, who is likely to be an analytically trained probabilist. Of course, where the proof is missing, there will be a reference to where it can be found in its full glory.

There are a number of appendices, and in particular, Appendices 1, 3, 5 and 6 give very heavily abridged accounts of those aspects of topology, differentiable manifolds, measures on locally compact spaces and compact operators in Hilbert space (respectively) that are needed. They can serve as a quick reminder of key facts, but are no substitute for systematic study of the topic in question. I have included references to standard texts for those who need them, but the reader can also find a fuller treatment of all of the above (and a great deal more, including a chapter on compact and locally compact groups, as well as representations of the former) in a two-volume work by Knapp [121, 122].

I hope the reader will find that the book contains a joyful and stimulating selection of topics. However one should be aware that compact groups are, though important, a restrictive class, and many important applications involve groups such as the Heisenberg group or the Euclidean group (the semi-direct product of translations with rotations). Analogues of some results obtained in this book are not yet available in these contexts, and there is a great deal of work still to be done. In particular, it is important that mathematicians, statisticians and engineers do not work in isolation as there is so much that they can learn from each other.

Guide to Notation and a Few Useful Facts

If S is a set, S^c denotes its complement. If T is another set then $S \backslash T := S \cap T^c$. If A is a finite set, then the number of elements in A is $\#A$. If A is a non-empty subset of a topological space then $\overline{A}$ is its closure.

$\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ are the sets of natural numbers, integers, rational numbers, real numbers and complex numbers (respectively). $\mathbb{Z}_+ := \mathbb{N} \cup \{0\}$. In this short section we use F to denote $\mathbb{R}$ or $\mathbb{C}$.

If S is a topological space, then the *Borel* σ-algebra of S will be denoted $\mathcal{B}(S)$. It is the smallest σ-algebra of subsets of S which contains all the open sets. Sets in $\mathcal{B}(S)$ are called *Borel sets*. $\mathbb{R}$ and $\mathbb{C}$ will always be assumed to be equipped with their Borel σ-algebras, and measurable functions from $(S, \mathcal{B}(S))$ to $(F, \mathcal{B}(F))$ are sometimes called *Borel measurable*. Similarly, a measure defined on $(S, \mathcal{B}(S))$ is called a *Borel measure*.

If S is a locally compact Hausdorff space, then $B_b(S, F)$ is the linear space (with the usual pointwise operations of addition and scalar multiplication) of all bounded Borel measurable functions from S to F. It is an F-Banach space under the supremum norm $||f||_\infty := \sup_{x \in S} |f(x)|$ for $f \in B_b(S, F)$. The space of bounded continuous functions from S to F is denoted $C_b(S, F)$. It is a closed linear subspace of $B_b(S, F)$, and so an F-Banach space in its own right. A function f from S to F is said to *vanish at infinity* if given any $\epsilon > 0$ there exists a compact set K in S so that $|f(x)| < \epsilon$ whenever $x \in K^c$. The space $C_0(S, F)$ of all continuous F-valued functions on S which vanish at infinity is a closed linear subspace of $B_b(S, F)$ (and of $C_b(S, F)$), and so is also an F-Banach space in its own right. The *support* of an F-valued function f defined on S is the closure of the set $\{x \in S; f(x) \neq 0\}$ and it is denoted $\text{supp}(f)$. The linear space $C_c(S, F)$ of all continuous F-valued functions on S with compact support is a dense subspace of $C_0(S, F)$. When $F = \mathbb{C}$, we usually write $B_b(S) := B_b(S, \mathbb{C})$, $C_0(S) := C_0(S, \mathbb{C})$ etc. This will be the case throughout the book, except for Chap. 5 where $C_0(S)$ always means $C_0(S, \mathbb{R})$ etc.

Throughout this book "smooth" means "infinitely differentiable". If M is a smooth manifold and $p \in \mathbb{N}$, then $C^p(M, F)$ is the linear space of all p-times continuously differentiable functions from M to F, and the linear space of infinitely differentiable functions is $C^\infty(M, F) = \bigcap_{p \in \mathbb{N}} C^p(M, F)$. We write $C(M, F) := C^0(M, F)$ for the linear space of all continuous functions on M. If M is compact, then $C(M, F) = C_b(M, F) = C_0(M, F) = C_c(M, F)$. Note that the linear space of smooth functions of compact support $C_c^\infty(M,F) = C^\infty(M,F) \cap C_c(M,F)$ is dense in $C_0(M, F)$. If $A(S)$ is any function space of mappings from S to F, then $A(S)_+$ denotes the cone of non-negative functions in $A(S)$.

If $n \in \mathbb{N}$, then $M_n(F)$ is the F-algebra of all $n \times n$ matrices with values in F. The identity matrix in $M_n(F)$ is denoted by I_n. If $A \in M_n(F)$, then A^T denotes its transpose and $A^* = \overline{A^T}$ is its adjoint. The trace of a square matrix A, i.e. the sum of its diagonal entries, is denoted by tr(A), and its determinant is $\det(A)$. The Hilbert–Schmidt inner product on $M_n(F)$ is $\langle A, B\rangle_{\text{HS}} := \text{tr}(\text{AB}^*)$, and the corresponding Hilbert–Schmidt norm is $||A||_{\text{HS}} := \text{tr}(\text{AA}^*)^{\frac{1}{2}}$. The matrix A is said to be *non-negative definite* if $x^TAx \geq 0$ for all $x \in \mathbb{R}^n$, and *positive definite* if $x^TAx > 0$ for all $x \in \mathbb{R}^n \setminus \{0\}$.

If (S, Σ, μ) is a measure space and $f : S \to F$ is an integrable function, we often write the Lebesgue integral $\int_S f(x)\mu(dx)$ as $\mu(f)$. For $1 \leq p < \infty$, $L^p(S) := L^p(S, \Sigma, \mu; F)$ is the usual L^p space of equivalence classes of functions that agree almost everywhere with respect to μ for which

$$||f||_p = \left(\int_S |f(x)|^p \mu(dx)\right)^{\frac{1}{p}} < \infty$$

for all $f \in L^p(S)$. $L^p(S)$ is a Banach space with respect to the norm $|| \cdot ||_p$, and $L^2(S)$ is a Hilbert space with respect to the inner product

$$\langle f, g\rangle := \int_S f(x)\overline{g(x)}\mu(dx)$$

for $f, g \in L^2(S)$. Once again, $L^p(S)$ will mean $L^p(S, \mathbb{C})$ in the main part of the book (except in Chap. 5).

The *indicator function* 1_A of $A \in \Sigma$ is defined as follows:

$$1_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A. \end{cases}$$

If μ is σ-finite, and v is a finite measure on (S, Σ), we write $v << \mu$ if v is absolutely continuous with respect to μ, and $\frac{dv}{d\mu}$ is the corresponding Radon–Nikodym derivative.

If (Ω, F, P) is a probability space and $X : \Omega \to \mathbb{R}$ is a random variable (i.e. a measurable function from (Ω, F) to $(\mathbb{R}, B(\mathbb{R}))$) that is also integrable in that $\int_\Omega |X(\omega)|P(d\omega) < \infty$, then its *expectation* is defined to be

$$\mathbb{E}(X) := \int_\Omega X(\omega)P(d\omega).$$

If $T : V_1 \to V_2$ is a linear mapping between F-vector spaces V_1 and V_2, then $\text{Ker}(T)$ is its kernel and $\text{Ran}(T)$ is its range. If $T : H_1 \to H_2$ is a bounded linear operator between F-Hilbert spaces H_i having inner products $\langle \cdot, \cdot \rangle_i \, (i = 1, 2)$, its *adjoint* is the unique bounded linear operator $T^* : H_2 \to H_1$ for which

$$\langle T^* \psi, \phi \rangle_1 = \langle \psi, T\phi \rangle_2,$$

for all $\phi \in H_1$, $\psi \in H_2$. The bounded linear operator $U : H_1 \to H_2$ is said to be *unitary* if it is both an isometry and a co-isometry (i.e. U^* is also an isometry). Equivalently it is an isometric isomorphism for which $U^{-1} = U^*$. If H is a F-Hilbert space, then $\mathcal{L}(H)$ will denote the $*$-algebra of all bounded linear operators on H, where the involution is obtained by taking adjoints. $\mathcal{L}(H)$ is a Banach space (and in fact a C^*-algebra, but we won't need this) with respect to the operator norm

$$||T|| = \sup\{||Tx||; x \in H, \, ||x|| = 1\}.$$

Note that the algebra $M_n(F)$ may be realised as $\mathcal{L}(F^n)$. More generally, if H_1 and H_2 are distinct F-Hilbert spaces, then $\mathcal{L}(H_1, H_2)$ will denote the F-linear space of all bounded linear operators from H_1 to H_2 . If H is a Hilbert space and $x, y \in H$ are orthogonal so that $\langle x, y \rangle = 0$, we sometimes write $x \perp y$. The *complexification* of a real vector space V is the complex vector space $V_{\mathbb{C}}$ whose elements are of the form $x + iy$, with $x, y \in V$, with the addition law:

$$(x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2),$$

for $x_i, y_i \in V (i = 1, 2)$ and scalar multiplication:

$$(\alpha + i\beta)(x + iy) = (\alpha x - \beta y) + i(\alpha y + \beta x),$$

for $x, y \in V$, $\alpha, \beta \in \mathbb{R}$. If V is equipped with an inner product $\langle \cdot, \cdot \rangle$, then we may define the inner product $\langle \cdot, \cdot \rangle_{\mathbb{C}}$ on $V_{\mathbb{C}}$ by the prescription

$$\langle x_1 + iy_1, x_2 + iy_2 \rangle_{\mathbb{C}} = \langle x_1, x_2 \rangle + \langle y_1, y_2 \rangle + i(\langle y_1, x_2 \rangle - \langle x_1, y_2 \rangle).$$

In the main part of the book, we will always write $\langle \cdot, \cdot \rangle_{\mathbb{C}}$ as $\langle \cdot, \cdot \rangle$. It is perhaps worth emphasising that all inner products on complex vector spaces are linear on the left and conjugate-linear on the right, which is standard in mathematics (but not in physics). The algebraic dual of an F-linear space V will typically be denoted by V^*, so V^* is the F-linear space of all F-linear functionals from V to F.

If T is a linear operator defined on an F-Banach space E, its domain is written as $\mathrm{Dom}(T)$. The *restriction* of T to $D \subset \mathrm{Dom}(T)$ is denoted $T|_D$. We say that T is *densely defined* if $\mathrm{Dom}(T)$ is dense in E and *closed* if its graph G_T is closed in $E \times E$, where $G_T := \{(\psi, T\psi); \psi \in \mathrm{Dom}\,(T)\}$. The linear operator T is said to be *closable* if it has at least one closed extension, in which case its *closure* $\overline{T}$ is the smallest such extension (i.e. that which has the smallest domain).

Let G be a group. We always denote its neutral element by e. General elements are typically denoted by g, h, but in the last part of the book we often use σ, τ. The group operation is usually written multiplicatively. If G is a Lie group its Lie algebra is denoted by $\mathfrak{g}$. Linear or unitary representations of groups are generically written as π, and the bounded operator $\pi(g)$ then acts on the complex Hilbert space V_π for each $g \in G$. The identity operator in V_π is denoted I_π. If V_π is finite-dimensional, then $d_\pi := \dim(V_\pi)$. The unitary dual of G is the set of all equivalence classes of irreducible unitary representations (with respect to unitary conjugation) denoted by $\widehat{G}$. This is precisely the dual group of G (i.e. the character group) when G is abelian. If G is a Lie group, the derived representation of its Lie algebra $\mathfrak{g}$ is $d\pi$. If G is a compact, connected Lie group, then the highest weight of π is denoted λ_π, its character is χ_π and the Casimir spectrum is $\{-\kappa_\pi, \pi \in \widehat{G}\}$. Generically we will denote characters by χ, weights by λ and roots by α. The Laplace–Beltrami operator associated to a Riemannian manifold is always written as Δ.

If G is a locally compact Hausdorff group, its left Haar measure is m_L and its right Haar measure is m_R. If G is also compact, then $m := m_L = m_R$, and we generally normalise so that $m(G) = 1$. If $f \in L^1(G, F)$, then $\int_G f(\sigma)m(d\sigma)$ is always written as $\int_G f(\sigma)d\sigma$. The Fourier transform of such an f at the representation π is $\widehat{f}(\pi) := \int_G \pi(g^{-1})f(g)dg$. Similarly, if μ is a finite Borel measure on G, then its Fourier transform at π is $\widehat{\mu}(\pi) := \int_G \pi(g^{-1})\mu(dg)$. Convolution of functions f_1 and f_2 is denoted as $f_1 * f_2$, and of measures μ_1 and μ_2 by $\mu_1 * \mu_2$.

We occasionally use Landau notation, according to which $(o(n), n \in \mathbb{N})$ is any real-valued sequence for which $\lim_{n\to\infty}(o(n)/n) = 0$ and $(O(n), n \in \mathbb{N})$ is any non-negative sequence for which $\limsup_{n\to\infty}(O(n)/n) < \infty$. Functions $o(t)$ and $O(t)$ are defined similarly. If f and g are functions from $\mathbb{R}$ to $\mathbb{R}$, then $f(x) \sim g(x)$ as $x \to \infty$ means that $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1$.

Is S is a countable set, the *Kronecker delta* is defined for $i, j \in S$ by

$$\delta_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j. \end{cases}$$

If $a, b \in \mathbb{R}$, then $a \wedge b := \min\{a, b\}$.