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Ralf Schindler

Set Theory

Exploring Independence and Truth

Springer

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To Julia, Gregor, and Joana, with love

Preface

Set theory aims at proving interesting true statements about the mathematical universe. Different people interpret "interesting" in different ways. It is well known that set theory comes from real analysis. This led to descriptive set theory, the study of properties of definable sets of reals, and it certainly is an important area of set theory. We now know that the theory of large cardinals is a twin of descriptive set theory. I find the interplay of large cardinals, inner models, and properties of definable sets of reals very interesting.

We give a complete account of the Solovay-Shelah Theorem according to which having all sets of reals to be Lebesgue measurable and having an inaccessible cardinal are equiconsistent. We give a modern account of the theory of $0^{\#}$, produce Jensen's Covering Lemma, and prove the Martin-Harrington Theorem according to which the existence of $0^{\#}$ is equivalent with Σ^1_1 determinacy. We also produce the Martin-Steel Theorem according to which Projective Determinacy follows from the existence of infinitely many Woodin cardinals.

I started learning logic by reading a script of my Master's thesis' advisor, Ulrich Blau, on a nude beach by the Ammersee near Munich back in 1989. It was a very enjoyable way of learning a fascinating and exciting subject, and I then decided to become a logician (In the meantime, Blau's script appeared as [6]). We shall assume in what follows that the reader has some familiarity with mathematical logic, to the extent of e.g. [11]. We are not going to explain the key concepts of first order logic.

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Berkeley, Girona, and Münster, February 2014 Ralf Schindler

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