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Lavinia Corina Ciungu

Non-commutative Multiple-Valued Logic Algebras

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Lavinia Corina Ciungu
Department of Mathematics
University of Iowa
Iowa City, IA, USA

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Dedicated to my precious son David Edward

Introduction

In 1920 Łukasiewicz introduced his three valued logic ([223]), the first model of multiple-valued logic. The n-valued propositional logic for $n > 3$ was constructed in 1922 and the $\aleph_0$-valued Łukasiewicz-Tarski logic in 1930 ([224]). The first completeness theorem for $\aleph_0$-valued Łukasiewicz-Tarski logic was given by Wajsberg in 1935. As a direct generalization of two-valued calculus, Post introduced in 1921 an n-valued propositional calculus distinct from that of Łukasiewicz ([239]).

In the early 1940s Gr.C. Moisil was the first to develop the theory of n-valued Łukasiewicz algebras with the intention of algebraizing Łukasiewicz's logic ([226, 227]), but an example of A. Rose from 1956 established that for $n \geq 5$ the Łukasiewicz implication can no longer be defined on a Łukasiewicz algebra. Consequently, the structures introduced by Moisil are models for Łukasiewicz logic only for $n = 3$ and $n = 4$. These algebras are now called *Łukasiewicz-Moisil algebras* or *LM algebras* for short ([14]).

The loss of implication has led to another type of logic, today called *Moisil logic*, distinct from the Łukasiewicz system. The logic corresponding to n-valued Łukasiewicz-Moisil algebras was created by Moisil in 1964. The fundamental concept of Moisil logic is *nuancing*. During 1954–1973 Moisil introduced the θ-valued LM algebras without negation, applied multiple-valued logics to switching theory and studied algebraic properties of LM algebras (representation, ideals, residuation) ([228]). Moisil's works have been continued by many mathematicians ([149, 151]). A. Iorgulescu introduced and studied θ-valued LM algebras with negation ([170]), while V. Boicescu defined and investigated n-valued LM algebras without negation ([13]).

Today these multiple-valued logics have been developed into fuzzy logics, which connect quantum mechanics, mathematical logic, probability theory, algebra and soft computing.

In 1958 Chang defined *MV-algebras* ([38]) as the algebraic counterpart of $\aleph_0$-valued Łukasiewicz logic and he gave another completeness proof of this logic ([39]).

An *MV-algebra* is an algebra $(A, \oplus, ^-, 0)$ with a binary operation $\oplus$, a unary operation $^-$ and a constant 0 satisfying the following equations:

(MV_1) $(x \oplus y) \oplus z = x \oplus (y \oplus z)$;
(MV_2) $x \oplus y = y \oplus x$;
(MV_3) $x \oplus 0 = x$;
(MV_4) $(x^-)^- = x$;
(MV_5) $x \oplus 0^- = 0^-$;
(MV_6) $(x^- \oplus y)^- \oplus y = (y^- \oplus x)^- \oplus x$.

Studies on MV-algebras have been developed in [5–8, 22, 77, 81, 87, 89, 91, 120, 139, 146, 147, 153, 213, 214, 217–219, 247].

Starting from the systems of positive implicational calculus, weak systems of positive implicational calculus and BCI and BCK systems, in 1966 Y. Imai and K. Iséki introduced the BCK-algebras ([168]).

In 1977 R. Grigolia introduced MV_n-algebras to model the n-valued Łukasiewicz logic ([157]) and it was proved that there is a connection between n-valued Łukasiewicz algebras and MV_n-algebras ([171–173, 191, 216]).

One of the most famous results in the theory of MV-algebras was Mundici's theorem from 1986 which states that the category of MV-algebras is equivalent to the category of Abelian ℓ-groups with strong unit ([229]).

The non-commutative generalizations of MV-algebras called *pseudo-MV algebras* were introduced by G. Georgescu and A. Iorgulescu in [135] and [137] and they can be regarded as algebraic semantics for a non-commutative generalization of a multiple-valued reasoning ([215]). The pseudo-MV algebras were introduced independently by J. Rachůnek ([241]) under the name of *generalized MV-algebras*.

A. Dvurečenskij proved in [97] that any pseudo-MV algebra is isomorphic with some interval in an ℓ-group with strong unit, that is, the category of pseudo-MV algebras is equivalent to the category of unital ℓ-groups.

Residuation is a fundamental concept of ordered structures and categories and Ward and Dilworth were the first to introduce the concept of a *residuated lattice* as a generalization of ideal lattices of rings ([262]). The theory of residuated lattices was used to develop algebraic counterparts of fuzzy logics ([256]) and substructural logics ([234]).

A residuated lattice is defined as an algebra $\mathcal{A} = (A, \wedge, \vee, \odot, \to, \rightsquigarrow, e)$ of type $(2, 2, 2, 2, 2, 0)$ satisfying the following conditions:

(A_1) $(A, \wedge, \vee)$ is a lattice;
(A_2) $(A, \odot, e)$ is a monoid;
(A_3) $x \odot y \leq z$ iff $x \leq y \to z$ iff $y \leq x \rightsquigarrow z$ for any $x, y, z \in A$ (*pseudo-residuation*).

A residuated lattice with a constant 0 (which can denote any element) is called a *pointed residuated lattice* or *full Lambek algebra* (*FL-algebra*, for short). If $x \leq e$ for all $x \in A$, then $\mathcal{A}$ is called an *integral residuated lattice*. An FL-algebra $\mathcal{A}$ which satisfies the condition $0 \leq x \leq e$ for all $x \in A$ is called *FL_w-algebra* or *bounded integral residuated lattice* ([129]). In this case we put $e = 1$, so that an FL_w-algebra will be denoted $(A, \wedge, \vee, \odot, \to, \rightsquigarrow, 0, 1)$. Clearly, if $\mathcal{A}$ is an FL_w-algebra, then $(A, \wedge, \vee, 0, 1)$ is a bounded lattice.

In order to formalize the multiple-valued logics induced by continuous t-norms on the real unit interval [0, 1], P. Hájek introduced in 1998 a very general multiple-valued logic, called *Basic Logic* (or BL) ([158]). Basic Logic turns out to be a common ingredient in three important multiple-valued logics: $\aleph_0$-valued Łukasiewicz logic, Gödel logic and Product logic. The Lindenbaum-Tarski algebras for Basic Logic are called *BL-algebras* ([23, 82, 220–222, 255–257]). Apart from their logical interest, BL-algebras have important algebraic properties and they have been intensively studied from an algebraic point of view.

The well-known result that a t-norm on $[0, 1]$ has residuum if and only if the t-norm is left-continuous makes clear that BL is not the most general t-norm based logic. In fact, a weaker logic than BL, called *Monoidal t-norm based logic* (MTL, for short) was defined in [117] and proved in [197] to be the logic of left-continuous t-norms and their residua. The algebraic counterpart of this logic is MTL-algebra, also introduced in [117].

G. Georgescu and A. Iorgulescu introduced in [136] the *pseudo-BL algebras* as a natural generalization of BL-algebras in the non-commutative case. A pseudo-BL algebra is an FL_w-algebra which satisfies the conditions:

(A_4) $(x \to y) \odot x = x \odot (x \rightsquigarrow y) = x \wedge y$ (*pseudo-divisibility*);
(A_5) $(x \to y) \vee (y \to x) = (x \rightsquigarrow y) \vee (y \rightsquigarrow x) = 1$ (*pseudo-prelinearity*).

Properties of pseudo-BL algebras were deeply investigated by A. Di Nola, G. Georgescu and A. Iorgulescu in [85] and [86]. Some classes of pseudo-BL algebras were investigated in [143] and the corresponding propositional logic was established by Hájek in [158] and [159].

A more general structure than the pseudo-BL algebra is the *weak pseudo-BL algebra* or *pseudo-MTL algebra* introduced by P. Flondor, G. Georgescu and A. Iorgulescu in [122]. Pseudo-MTL algebras are FL_w-algebras satisfying condition (A_5) and they include as a particular case the *weak BL-algebras* which is an alternative name for MTL-algebras.

Properties of pseudo-MTL algebras are also studied in [46, 144, 181].

An FL_w-algebra which satisfies condition (A_4) is called a *divisible residuated lattice* or *bounded Rℓ-monoid*. Properties of divisible residuated lattices were studied by A. Dvurečenskij, J. Rachůnek and J. Kühr ([105, 111, 205, 240]).

Pseudo-BCK algebras were introduced in 2001 by G. Georgescu and A. Iorgulescu ([138]) as non-commutative generalizations of BCK-algebras. Properties of pseudo-BCK algebras and their connection with other fuzzy structures were established by A. Iorgulescu in [179–182].

For a guide through the pseudo-BCK algebras realm we refer the reader to the monograph [186].

Another generalization of pseudo-BL algebras was given in [148], where *pseudo-hoops* were defined and studied. Pseudo-hoops were originally introduced by Bosbach in [15] and [16] under the name of *complementary semigroups*. It was proved that a pseudo-hoop has the pseudo-divisibility condition and it is a meet-semilattice, so a bounded Rℓ-monoid can be viewed as a bounded pseudo-hoop together with the join-semilattice property. In other words, a bounded pseudo-hoop is a meet-semilattice ordered residuated, integral and divisible monoid.

Other topics in multiple-valued logic algebras have been studied in [34, 36, 92, 132, 141, 150, 248].

The notion of a *state* is an analogue of a probability measure and it has a very important role in the theory of quantum structures ([108]). The basic idea of states is an averaging of events (elements) of a given algebraic structure. Since in the case of Łukasiewicz ∞-valued logic the set of events has the structure of an MV-algebra, the theory of probability on this logic is based on the notion of a state defined on an MV-algebra. Besides mathematical logic, Riečan and Neubrunn studied MV-algebras as fields of events in generalized probability theory ([250]). Therefore, the study of states on MV-algebras is a very active field of research ([40, 83, 84, 119, 133, 246]) which arises from the general problem of investigating probabilities defined for logical systems.

States on an MV-algebra $(A, \oplus, ^-, 0)$ were first introduced by D. Mundici in [230] as functions $s : A \longrightarrow [0, 1]$ satisfying the conditions:

$s(1) = 1$ (*normality*);
$s(x \oplus y) = s(x) + s(y)$ if $x \odot y = 0$ (*additivity*),

where $x \odot y = (x^- \oplus y^-)^-$.

They are analogous to finitely additive probability measures on Boolean algebras and play a crucial role in MV-algebraic probability theory ([249]).

States on other commutative and non-commutative algebraic structures have been defined and investigated by many authors ([20, 21, 102, 133, 134, 140, 142, 258, 259]).

The aim of this book is to present new results regarding non-commutative multiple-valued logic algebras and some of their applications. Almost all the results are based on the author's recent papers ([42–75]).

The book consists of nine chapters.

The Chap. 1 is devoted to pseudo-BCK algebras. After presenting the basic definitions and properties, we prove new properties of pseudo-BCK algebras with pseudo-product and pseudo-BCK algebras with pseudo-double negation. Examples of proper pseudo-BCK algebras, good pseudo-BCK algebras and pseudo-BCK lattices are given, and the orthogonal elements in a pseudo-BCK algebra are characterized. Finally, we define the maximal and normal deductive systems of a pseudo-BCK algebra with pseudo-product and we study their properties.

In Chap. 2 we recall the basic properties of pseudo-hoops, we introduce the notions of join-center and cancellative-center of pseudo-hoops and we define and study algebras on subintervals of pseudo-hoops. Additionally, new properties of a pseudo-hoop are proved.

Chapter 3 is devoted to residuated lattices. We investigate the properties of the Boolean center of an FL_w-algebra and we define and study the directly indecomposable FL_w-algebras. One of the main results consists of proving that any linearly ordered FL_w-algebra is directly indecomposable. Finally, we define and study FL_w-algebras of fractions relative to a meet-closed system.

In Chap. 4 we present some specific properties of other non-commutative multiple-valued logic algebras: pseudo-MTL algebras, bounded Rℓ-monoids, pseudo-BL algebras and pseudo-MV algebras. As main results, we extend to the case of pseudo-MTL algebras some results regarding prime filters proved for pseudo-BL algebras. The Glivenko property for a good pseudo-BCK algebra is defined and it is shown that a good pseudo-hoop has the Glivenko property.

Chapter 5 deals with special classes of non-commutative residuated structures: local, perfect and Archimedean structures. The local bounded pseudo-BCK(pP) algebras are characterized in terms of primary deductive systems, while the perfect pseudo-BCK(pP) algebras are characterized in terms of perfect deductive systems. One of the main results consists of proving that the radical of a bounded pseudo-BCK(pP) algebra is a normal deductive system. We also prove that any linearly ordered pseudo-BCK(pP) algebra and any locally finite pseudo-BCK(pP) algebra are local. Other results state that any local FL_w-algebra and any locally finite FL_w-algebra are directly indecomposable. The classes of Archimedean and hyperarchimedean FL_w-algebras are introduced and it is proved that any locally finite FL_w-algebra is hyperarchimedean and any hyperarchimedean FL_w-algebra is Archimedean.

Chapter 6 is devoted to the presentation of states on multiple-valued logic algebras. We introduce the notion of states on pseudo-BCK algebras and we study their properties. One of the main results consists of proving that any Bosbach state on a good pseudo-BCK algebra is a Riečan state, however the converse turns out not to be true. We also prove that every Riečan state on a good pseudo-BCK algebra with pseudo-double negation is a Bosbach state. In contrast to the case of pseudo-BL algebras, we show that there exist linearly ordered pseudo-BCK algebras having no Bosbach states and that there exist pseudo-BCK algebras having normal filters which are maximal, but having no Bosbach states.

Some specific properties of states on FL_w-algebras, pseudo-MTL algebras, bounded Rℓ-monoids and subinterval algebras of pseudo-hoops are proved.

A special section is dedicated to the existence of states on the residuated structures, showing that every perfect FL_w-algebra admits at least a Bosbach state and every perfect pseudo-BL algebra has a unique state-morphism.

Finally, we introduce the notion of a local state on a perfect pseudo-MTL algebra and we prove that every local state can be extended to a Riečan state.

In Chap. 7 we generalize measures on BCK algebras introduced by A. Dvurečenskij in [94] and [108] to pseudo-BCK algebras that are not necessarily bounded. In particular, we show that if A is a downwards-directed pseudo-BCK algebra and m a measure on it, then the quotient over the kernel of m can be embedded into the negative cone of an Abelian, Archimedean ℓ-group as its subalgebra. This result will enable us to characterize nonzero measure-morphisms on downwards-directed pseudo-BCK algebras as measures whose kernel is a maximal filter. We study state-measures on pseudo-BCK algebras with strong unit and we show how to characterize state-measure-morphisms as extremal state-measures or as state-measures whose kernel is a maximal filter. In particular, we show that for unital pseudo-BCK algebras that are downwards-directed, the quotient over the kernel can be embedded into the negative cone of an Abelian, Archimedean ℓ-group with strong unit. We generalize to pseudo-BCK algebras the identity between de Finetti maps and Bosbach states, following the results proved by Kühr and Mundici in [211] who showed that de Finetti's coherence principle, which has its origins in Dutch bookmaking, has a strong relationship with MV-states on MV-algebras. We also generalize this for state-measures on unital pseudo-BCK algebras that are downwards-directed.

Chapter 8 is devoted to generalized states on residuated structures. The study of these generalized states is motivated by their interpretation as a new type of semantics for non-commutative fuzzy logics. Usually, the truth degree of sentences in a fuzzy logic is a number in the interval $[0, 1]$ or, more generally, an element of an FL_w-algebra. Similarly, for generalized states, the probability of sentences is evaluated in an arbitrary FL_w-algebra.

We define the generalized states of type I and type II and generalized state-morphisms and we study the relationship between them. We prove that any perfect FL_w-algebra admits strong type I and type II states. Some conditions are given for a generalized state of type I on a linearly ordered bounded $\mathrm{R}\ell$-monoid to be a state operator. The notion of a strong perfect FL_w-algebra is introduced and it is proved that any strong perfect FL_w-algebra admits a generalized state-morphism. The notion of a generalized Riečan state is also introduced and the main results are proved based on the Glivenko property defined for the non-commutative case. The main results consist of proving that any order-preserving type I state is a generalized Riečan state and in some particular conditions the two states coincide. We introduce the notion of a generalized local state on a perfect pseudo-MTL algebra A and we prove that, if A is relatively free of zero divisors, then every generalized local state can be extended to a generalized Riečan state.

Chapter 9 deals with residuated structures with internal states. We define the notions of state operator, strong state operator, state-morphism operator, weak state-morphism operator and we study their properties. We prove that every strong state pseudo-hoop is a state pseudo-hoop and any state operator on an idempotent pseudo-hoop is a weak state-morphism operator. It is proved that for an idempotent pseudo-hoop A a state operator on $\mathrm{Reg}(A)$ can be extended to a state operator on A. One of the main results of this chapter consists of proving that every perfect pseudo-hoop admits a nontrivial state operator. Other results compare the state operators with states and generalized states on a pseudo-hoop. Some conditions are given for a state operator to be a generalized state and for a generalized state to be a state operator.

We hope that this book will be useful to graduate students and researchers in the area of algebras of multiple-valued logics.

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Iowa City, USA Lavinia Corina Ciungu
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