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Christopher S. Hardin • Alan D. Taylor

The Mathematics of Coordinated Inference

A Study of Generalized Hat Problems

 Springer

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To Novem

C.S.H.

To Joel and Alan

A.D.T.

Preface

This book deals with the question of how successfully one can predict the value of an arbitrary function at one or more points of its domain based on some knowledge of its values at other points. In large part because of the axiom of choice, the degree of success turns out to be quite remarkable in a number of different situations. For example, there is a method of predicting the value $f(a)$ of a function f mapping the reals to the reals, based only on knowledge of f 's values on the interval $(a - 1, a)$, and for every such function the prediction is incorrect only on a countable set that is nowhere dense. In many cases we are able to show that there is a “predictor” that is as successful as possible, and that it is essentially unique.

We collect together most of what is known regarding this problem and we provide a number of extensions of the results that we have published in a series of seven papers over the past six years. We also indicate a number of open questions, ranging from the combinatorially difficult finite to those arising from some results that we show are independent of ZFC plus a fixed value of the continuum.

New York, NY, USA
Schenectady, NY, USA

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