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An Introduction to the Kähler–Ricci Flow

 Springer

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ISBN 978-3-319-00818-9 ISBN 978-3-319-00819-6 (eBook)
DOI 10.1007/978-3-319-00819-6
Springer Cham Heidelberg New York Dordrecht London

Lecture Notes in Mathematics ISSN print edition: 0075-8434
ISSN electronic edition: 1617-9692

Library of Congress Control Number: 2013946026

Mathematics Subject Classification (2010): 35K96, 32Q15, 32Q20, 32Q25, 14E30

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Printed on acid-free paper

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Preface

In 2010–2011 we organized several meetings of our ANR project MACK. These consisted of series of lectures centered around the Kähler–Ricci flow, which took place, respectively, in

- IMT (Toulouse, France), February 2010: mini course by H.-D. Cao:
An introduction to the Kähler–Ricci flow on Fano manifolds;
- LATP (Marseille, France), March 2010: mini course by J. Song:
The Kähler–Ricci flow on complex surfaces;
- CIRM (Luminy, France), February 2011: mini course B. Weinkove:
An introduction to the Kähler–Ricci flow;
- CIRM (Luminy, France), February 2011: mini course J. Song:
Kähler–Ricci flow and the Minimal Model Program;
- IMT (Toulouse, France), June 2011: mini course by D.-H. Phong:
The normalized Kähler–Ricci flow on Fano manifolds;
- IMT (Toulouse, France), June 2011: mini course by S. Boucksom and V. Guedj:
Regularizing properties of the Kähler–Ricci flow;
- FSSM (Marrakech, Morocco), October 2011: mini course by C. Imbert:
Introduction to fully nonlinear parabolic equations;
- FSSM (Marrakech, Morocco), October 2011: mini course by V. Guedj:
Convergence of the Kähler–Ricci flow on Kähler–Einstein Fano manifolds.

There were other lectures on more algebraic aspects (e.g., an introduction to the Minimal Model Program and finite generation of the canonical ring by S. Druel at the CIRM), or on elliptic problems (e.g., Moser–Trudinger inequalities by B. Berndtsson in Marrakech). Some of the speakers have produced a set of lecture notes, working hard to make them accessible to non-experts. This volume presents them in a unified way.

It is a pleasure to thank all the participants of these meetings for their enthusiasm and for creating a very pleasant atmosphere of work. Special thanks of course to the following lecturers:

- Sébastien Boucksom (CNRS and IMJ, Paris, France);
- Huai Dong Cao (Lehigh University, Bethlehem, USA);
- Vincent Guedj (IUF and IMT, Toulouse, France)
- Cyril Imbert (CNRS and Université Paris-Est Créteil, France);
- Duong Hong Phong (Columbia University, USA);
- Jian Song (Rutgers University, Piscataway, USA);
- Ben Weinkove (University of California, San Diego, USA).

We acknowledge financial support from the French ANR project MACK. We also would like to thank the CIRM and their staff for providing wonderful conditions of work during the thematic month “Complex and Riemannian geometry,” as well as the LATP and the IMT for providing “professeur invité” positions (resp. for H.-D. Cao, J. Song, and D.-H. Phong).

Our last meeting in Marrakech was also extremely useful, we thank the organizers (Said Asserda and Ahmed Zeriahi), the other speakers (B. Berndtsson, S. Boucksom, A. Broustet, J.-P. Demailly, S. Diverio, N.C. Nguyen, S. Lamy) as well as all the participants for their help in creating a successful event.

Paris, France
Grenoble, France
Toulouse, France
1 July 2012

S. Boucksom
P. Eyssidieux
V. Guedj

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