

Lyapunov Functionals and Stability of Stochastic Functional Differential Equations

Leonid Shaikhet

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Preface

This book deals with stability of stochastic functional differential equations and continues and complements the previous book of the author, “Lyapunov Functionals and Stability of Stochastic Difference Equations” [278].

Functional differential equations (also called hereditary systems [9], or systems with aftereffect [140], or equations with memory [11], or equations with deviating arguments [66, 167], or equations with delays [10, 24–29, 54, 93, 97, 226], or equations with time lag [65, 72, 103], or retarded differential equations [52, 78], or differential difference equations [22]) are infinite-dimensional ones [88, 181], contrary to ordinary differential equations, and describe the processes whose behavior depends not only on their present state but also on their past history [104–106, 131–133]. Systems of such type are widely used to model processes in physics, mechanics, automatic regulation, economy, finance, biology, ecology, sociology, medicine, etc. (see, e.g., [15, 18, 20, 24–26, 32, 36–41, 57, 58, 71, 91, 92, 98, 160, 161, 165, 177, 186, 189, 192, 199, 212, 213, 218, 219, 224, 245, 253, 257, 286, 287, 293, 303]).

The first mathematical models with functional differential equations have been studied during the 18th century (L. Euler, J. Bernoulli, M.J. Condorcet, J. Lagrange, P. Laplace). In the beginning of 20th century the development of the delay systems study started from the works by Vito Volterra [302], linked to viscoelasticity and ecology. This pioneer work was continued by Tsytkin [298–300], Myshkis [220, 221], Kac and Krasovskii [120, 156–158], Elsgoltz [66], Razumikhin [239–242], and many other (see, for example, [5, 9, 22, 43, 59, 62, 67, 100, 103–106, 111, 126, 132, 133, 140, 194, 206, 308, 313]).

An important direction in the study of hereditary systems is their stability [3, 12, 55, 56, 88, 96, 103, 125, 134, 135, 181, 182, 191, 195–197, 204–207, 211, 215–217, 231, 236, 258–260, 301]. As it was proposed by Krasovskii in the 1950s [156–159], a stability condition for differential equation with delays can be obtained using an appropriate Lyapunov functional. The construction of different Lyapunov functionals for one differential equation with delay allows one to get different stability conditions for the solution of this equation. The method of Lyapunov–Krasovskii functionals is very popular and developing until now [78, 79]. However, the construction of each Lyapunov functional required a unique work from its author. In

1975, Shaikhet [261] introduced a parametric family of Lyapunov functionals, so that an infinite number of Lyapunov functionals were used simultaneously. This approach allowed one to get different stability conditions for the considered equation using only one Lyapunov functional.

During twenty last years, a general method of constructing Lyapunov functionals was proposed and developed by Kolmanovskii and Shaikhet for stochastic functional differential equations, for stochastic difference equations with discrete time and continuous time, and for partial differential equations [48, 130, 131, 136–139, 145–149, 265, 269–272, 278, 280, 281]. This method was successfully applied to stability research of some mathematical models in mechanics, biology, ecology, etc. (see, for instance, [27, 36–39, 267, 273, 276]). Nevertheless, it should be noted that the stability theory for stochastic hereditary system dynamically develops and has yet a number of unsolved problems [277, 279].

Usually the books devoted to the stability theory for functional differential equations do not concern numerical determination of stability domains or the behavior of the solutions. In spite of that, this book, along with the modern theoretical results, includes also many numerical investigations showing both the stability domains and structure of the solutions. It offers a certain amount of analytical mathematics, practical numerical procedures, and actual implementations of these approaches.

In this book, consisting of twelve chapters, a general method of construction of Lyapunov functionals for stochastic functional differential equations is expounded.

Introductory Chap. 1 presents general classification and some peculiarities of functional differential equations, some properties of their solutions, the method of steps and the characteristic equation for retarded differential equations, the dependence of solution stability on small delay in equation, and the Routh–Hurwitz stability conditions for systems without delay. This section covers some theoretical backgrounds of the differential equations used in the book with concentration on mathematical rigor.

In Chap. 2 short introduction to stochastic functional differential equations is presented, in particular, the definition of the Wiener process and its numerical simulation, the Itô integral, and the Itô formula. Different definitions of stability for stochastic functional differential equations are also considered, basic Lyapunov-type stability theorems, and description of the procedure of constructing Lyapunov functionals for stability investigation. In this section some useful statements, some useful inequalities, and some unsolved problems are also included.

In Chap. 3 the procedure of constructing Lyapunov functionals is used to obtain conditions for stability of scalar stochastic linear delay differential equations with constant and variable coefficients and with constant and variable delays. It is shown that different ways of constructing Lyapunov functionals for a given equation allow us to get different conditions for asymptotic mean-square stability of the zero solution of this equation.

In Chap. 4 the procedure of constructing Lyapunov functionals is demonstrated for stability investigation of stochastic linear systems of two equations with constant and distributed delays and with constant and variable coefficients.

In Chap. 5 the stability of the zero solution and positive equilibrium points for nonlinear systems is studied. In particular, differential equations with nonlinearities

in deterministic and stochastic parts and with fractional nonlinearity are considered. It is shown that investigation of stability in probability for nonlinear systems with the level of nonlinearity higher than one can be reduced to investigation of the asymptotic mean-square stability of the linear part of the considered nonlinear system.

In Chap. 6 the general method of construction of Lyapunov functionals is used to get the asymptotic mean-square stability conditions for stochastic linear differential equations with constant delay, with distributed delay, and with variable bounded and unbounded delays. Sufficient stability conditions are formulated in terms of the existence of positive definite solutions of some matrix Riccati equations. Using the procedure of constructing Lyapunov functionals, it is shown that for one stochastic linear differential equation, several different matrix Riccati equations can be obtained that allow one to get different stability conditions.

In Chap. 7 sufficient conditions for asymptotic mean-square stability of the solutions of stochastic differential equations with delay and Markovian switching are obtained. Taking into account that it is difficult enough in each case to get analytical stability conditions, a numerical procedure for investigation of stability of stochastic systems with Markovian switching is considered. This procedure can be used in the cases where analytical conditions of stability are absent. Some examples of using the proposed numerical procedure are considered. Results of the calculations are presented by a lot of figures.

Chapter 8 is devoted to the classical problem of stabilization of the controlled inverted pendulum. The problem of stabilization for the mathematical model of the controlled inverted pendulum during many years is very popular among the researchers. Unlike the classical way of stabilization in which the stabilized control is a linear combination of the states and velocities of the pendulum, here another way of stabilization is proposed. It is supposed that only the trajectory of the pendulum can be observed and stabilized control depends on the whole trajectory of the pendulum. Via the general method of construction of Lyapunov functionals, sufficient conditions for stabilization by stochastic perturbations are obtained, and nonzero steady-state solutions are investigated.

In Chap. 9 the well-known Nicholson blowflies equation with stochastic perturbations is considered. Sufficient conditions for stability in probability of the trivial and positive equilibrium points of this nonlinear differential equation with delay are obtained.

In Chap. 10 the mathematical model of the type of predator–prey with aftereffect and stochastic perturbations is considered. Sufficient conditions for stability in probability of the positive equilibrium point of the considered nonlinear system are obtained.

Chapter 11 deals with a mathematical model of the spread of infectious diseases, the so-called SIR epidemic model. Sufficient conditions for stability in probability of two equilibrium points of the SIR epidemic model with distributed delays and stochastic perturbations are obtained.

In Chap. 12 mathematical models are considered that describe human behaviors related to some addictions: consumption of alcohol and obesity. The existence of

positive equilibrium points for these models are shown, and sufficient conditions for stability in probability of these equilibrium points are obtained.

The bibliography at the end of the book does not pretend to be complete and includes some of the author's publications [261–279], his publications jointly with coauthors [27, 36–39, 48, 64, 77, 136–148, 155, 197, 232, 246, 280, 281], and the literature used by the author during preparation of this book.

The book is addressed both to experts in stability theory and to a wider audience of professionals and students in pure and computational mathematics, physics, engineering, biology, and so on.

The book is mostly based on the results obtained by the author independently or jointly with coauthors, in particular, with the friend and colleague V. Kolmanovskii, with whom the author is glad and happy to collaborate for more than 30 years.

Taking into account that the possibilities for further improvement and development are endless, the author will appreciate receiving useful remarks, comments, and suggestions.

Donetsk, Ukraine

Leonid Shaikhet

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