Part IV
Boundary-Value Problems on a Half-Line

Introduction to Part IV

We now return to the study of well-posedness of boundary-value problems. In Chapter 3, we investigated well-posedness of the Cauchy problem for equation (1) on the half-line $R_+ = [0, +\infty)$ as well as on a finite segment $[0, T]$. In Chapter 4, we studied well-posedness of the Dirichlet and Neumann problems for equation (1) on a finite segment $[0, T]$.

In the present part, we consider well-posedness of the Dirichlet and Neumann problems for equation (1) on the half-line R_+.

Definition 11.1. Let A and B be c.n.o. in H, $f \in H$. A solution $y(t)$ of equation (1) on R_+ such that $y(0) = f$ ($y'(0) = f$) is said to be a solution of the Dirichlet (Neumann) problem for (1) on R_+ with boundary data f.

Definition 11.2. Let A and B be c.n.o. in H, $f \in H$ ($f \in \Phi'$). A weak solution $y(t)$ of equation (1) on R_+ such that $y(0) = f$ ($(y(t), g)'|_{t=0} = (f, g)$, $\forall g \in \Phi$) is said to be a weak solution of the Dirichlet (Neumann) problem for (1) on R_+ with boundary data f.

In Chapter 11, we consider the Dirichlet problem for equation (1) on R_+. The situation for this problem on $[0, +\infty)$ differs essentially from the situation for this problem on a finite segment. Let us explain this.

Let A and B be arbitrary c.n.o. in H. A boundary condition $y(0) = f_0$ does not single out a unique solution (weak solution) of equation (1) on R_+. Indeed, let $f_0 = 0$. For any $k \geq 1$ and any $f_1 \in E_k H : y(t) = \psi_1(A, B, t) f_1 (t \in R_+)$ is a solution of (1) on R_+ such that $y(0) = 0$.

We obtain that in the space $C(R_+, H)$ there is more than one solution of equation (1) on R_+ such that $y(0) = 0$.

For the condition $y(0) = 0$ to single out a unique solution (weak solution) $y(t)$ of equation (1) on R_+, we seek this solution (weak solution) in a linear subspace of $C(R_+, H)$ which is narrower than the whole $C(R_+, H)$. In this narrower space, a solution (weak solution) $y(t)$ of (1) on R_+ such that $y(0) = 0$ may be unique.

Definition 11.3. Let A and B be c.n.o. in H. A linear subspace of $C(R_+, H)$ is said to be a class of uniqueness (weak uniqueness) for the Dirichlet problem for equation (1) on R_+ if there exists a unique solution (weak solution) $y(t)$ of (1) on R_+ from this subspace such that $y(0) = 0$.

Let A and B be arbitrary c.n.o. in H. The linear subspace of $C(R_+, H)$ which consists of the only function $y(t) = 0$ $(t \in R_+)$ is a class of (weak) uniqueness for the Dirichlet problem for equation (1) on R_+. The whole $C(R_+, H)$ is not a class of (weak) uniqueness for the Dirichlet problem for equation (1) on R_+.

In general, one and the same linear subspace of $C(R_+, H)$ may be a class of (weak) uniqueness for the Dirichlet problem for a certain equation (1) on R_+ but not be a class of (weak) uniqueness for another equation.

We consider different linear subspaces of $C(R_+, H)$. Each of these subspaces is defined by a certain condition on behaviour of functions at infinity.

Definition 11.4. Let $\gamma \in R^1$. Denote by Y_γ the space of all $y(t) \in C(R_+, H)$ such that $e^{-\gamma t} \parallel y(t) \parallel \to 0$ as $t \to +\infty$ (i.e., $\exists C_y(t) \in C(R_+)$ such that $C_y(t) \to 0$ as $t \to +\infty$ and $\parallel y(t) \parallel \leq C_y(t) e^{\gamma t}$, $\forall t \in R_+$).

For instance, if $\gamma = 0$ then Y_γ is the space of all functions $y(t) \in C(R_+, H)$ such that $y(t)$ converges to 0 in H as $t \to +\infty$.

For each of these linear spaces Y_γ, we answer the following question: to indicate *all* pairs of c.n.o. A and B in H such that this space is a class of (weak) uniqueness for the Dirichlet problem for equation (1) on R_+.

This subspace of $C(R_+, H)$ corresponds to a certain condition on behaviour of functions at infinity. In fact, we indicate *all* equations (1) with c.n.o. A and B in H such that there exists a unique (weak) solution $y(t)$ of (1) on R_+ with $y(0) = 0$ which satisfies this condition on behaviour at infinity.

Starting from these results, we consider (weak) well-posedness of the Dirichlet problem for equation (1) on R_+.

Definition 11.5. Let A and B be c.n.o. in H, U be a linear subspace of $C(R_+, H)$, and G be a linear subspace of H. We shall say that the Dirichlet problem in the class of functions U for equation (1) on R_+ is (weakly) well-posed in the space of boundary data G if for each $f \in G$ there exists a unique (weak) solution of equation (1) on R_+ from U such that $y(0) = f$.

Obviously, Definition 11.5 holds if and only if simultaneously:

1) U is a class of (weak) uniqueness for the Dirichlet problem for equation (1) on R_+;
2) for each $f \in G$ there exists a (weak) solution of equation (1) on R_+ from U with $y(0) = f$.

For each of the indicated linear subspaces of $C(R_+, H)$, we give a condition on c.n.o. A and B in H which is necessary and sufficient for the Dirichlet problem in this space of functions for equation (1) on R_+ to be weakly well-posed in $G=H$. Similarly, we consider well-posedness of the Dirichlet problem in $G= D(B) \cap D(A^2)$.

The Neumann problem for equation (1) on R_+ is treated analogously in Chapter 12. One may consider the Dirichlet and Neumann problems for equation (1) on $(-\infty, 0]$ quite similarly (cf. Section 6 of Chapter 4).