Part II
Initial Data of Solutions

Introduction to Part II

In Sections 3, 5 of Chapter 4, we consider the following question. Let A and B be c.n.o. in H. We fix a certain boundary-value problem for equation (1) on $[a,b]$ and a certain space of boundary data $G \subseteq H \times H (G \subseteq \Phi' \times \Phi')$, which may depend on A and B. Further, we assume that this boundary-value problem is well-posed (weakly well-posed) in G, i.e., for any $(f_0, f_1) \in G$ there exists a unique solution (weak solution) of this boundary-value problem for equation (1) on $[a,b]$ with b.d. (f_0, f_1). The question is: what is the set of *all* (f_0, f_1) such that this boundary-value problem for equation (1) on $[a,b]$ with b.d. (f_0, f_1) has a unique solution (weak solution)? (It is obvious that this set contains G.)

We have studied this question for the Dirichlet and Neumann problems. In the second part of the work, we go on to study this question for the Cauchy problem (and, consequently, for the inverse Cauchy problem, since for that problem all the results are preserved with the replacements indicated in Section 6 of Chapter 4).

We treat all the questions for $R_+ = [0, +\infty)$. For any other $[a,b]$ such that $-\infty < a < b \le +\infty$, all the statements remain valid.

Denote for each $(\lambda, \mu) \in C^2 : a = -\frac{1}{2}\operatorname{Re}(\omega_1 + \omega_2) = \frac{1}{2}\operatorname{Re}\lambda$, $b_1 = \frac{1}{2}|\operatorname{Re}(\omega_1 - \omega_2)| = |\operatorname{Re}\sqrt{\frac{\lambda^2}{4} - \mu}|$, $b_2 = \frac{1}{2}|\operatorname{Im}(\omega_1 - \omega_2)| = |\operatorname{Im}\sqrt{\frac{\lambda^2}{4} - \mu}|$, $d = -\max_{i=1,2}\operatorname{Re}\omega_i = a - b_1$. Taking into account Theorem 3.1, Lemmas 3.2 and 3.3, Theorem 3.2, and Statement 3.4, we obtain the following statement.

Statement 5.1. Let A and B be c.n.o. in H, and let $P = \sigma(A, B)$. The following conditions are equivalent:

1) the Cauchy problem for equation (1) on R_+ is strongly well-posed in the sense of H.O. Fattorini's definition (Section 7 of Chapter 3; [93, Chapter VIII]);
2) the Cauchy problem for (1) on R_+ is well-posed in $G = D(B) \times H_1 = D(B) \times (D(A) \cap D(|B|^{1/2}))$;
3) the Cauchy problem for (1) on R_+ is weakly well-posed in $G = H \times H_{-1} = H \times H_-(|A| + |B|^{1/2} + 1)$;
4) $\exists 0 \le \gamma, C < +\infty : P \subseteq \{(\lambda, \mu) \in C^2 | d \ge -\gamma, |\operatorname{Im}\lambda|^2 \le C(1 + a^2 + b_1^2 + b_2^2)\}$. (2)

Here (2) for $P = \sigma(A,B)$ means that both conditions (2.2) (i.e., (3.3)) and (3.4) hold.

Let A and B be c.n.o. in H such that $P = \sigma(A,B)$ satisfies (2).

The Cauchy problem for equation (1) on R_+ is well-posed in $G = D(B)\times H_1$, i.e., for any $(f_0, f_1) \in D(B) \times (D(A) \cap D(|B|^{1/2}))$: the Cauchy problem for (1) on R_+ with initial conditions (i.c.) $y(0) = f_0$, $y'(0) = f_1$ has a unique solution. Now a natural question arises:

1. What is the set of *all* (f_0, f_1) such that the Cauchy problem for (1) on R_+ with i.c. $y(0) = f_0$, $y'(0) = f_1$ has a unique solution?

We denote this set by F_C. By virtue of Definition 1.1, for every solution $y(t)$ of (1) on $R_+ : y(0) \in D(B)$, $y'(0) \in D(A)$. Taking into account Statement 5.1, we have:

$$D(B) \times (D(A) \cap D(|B|^{1/2})) \subseteq F_C \subseteq D(B) \times D(A).$$

Thus, $F_C \subseteq D(B) \times D(A) \subseteq H \times H_{-1}$. But Statement 5.1 implies that for every $(f_0, f_1) \in H \times H_{-1}$ there exists a unique weak solution of the Cauchy problem for (1) on R_+ with b.d. (f_0, f_1). So the question 1 is:

– for what $(f_0, f_1) \in H \times H_{-1}$ this weak solution is *usual*?

Since $F_C \subseteq H \times H_{-1}$, we conclude from Statement 5.1 that there exists a one-to-one mapping of F_C onto the set Y of all solutions of (1) on R_+; this mapping is given by the formula: $Y \ni y(t) \leftrightarrow (y(0), y'(0)) \in F_C$. Therefore, the question 1 may also be considered as the following one:

– to indicate *all* solutions of equation (1) on R_+.

In order to answer this question, we need to introduce a new integral transform $R(t)$ $(t \geq 0)$ for σ-finite measures and to study its boundary behaviour as $t \to 0$ depending on the sub-integral measure. Let us explain this in more detail.

At first, let us say some words on definitions. Generally speaking, a measure ρ defined on Borel subsets of a closed $P \subseteq C^2$ is said to be σ-finite if there exists a sequence $\{P_k\}_{k=1}^{\infty}$ such that $P = \bigcup_{k=1}^{\infty} P_k$ and $\rho(P_k) < +\infty$ for all $k \geq 1$. In this monograph, we use the term «σ-finite» in a more restricted sense. Namely, we demand that as this sequence one can take $P_k = P \cap \square_k = \{(\lambda, \mu) \in P \| \lambda| \leq k, |\mu| \leq k\}$, $k \geq 1$. In other words, we call a measure ρ σ-finite if $\rho(\Delta) < +\infty$ for all bounded Borel sets $\Delta \subseteq P$.

Let's attempt to answer the question 1. By Theorem 3.1, for any $(f_0, f_1) \in H \times H_{-1}$: if there exists a solution of (1) on R_+ such that $y(0) = f_0$, $y'(0) = f_1$, then it is unique and has the form: $y(t) = \psi_0(A,B,t)f_0 + \psi_1(A,B,t)f_1$ $(t \in R_+)$. Thus, we have to indicate all $(f_0, f_1) \in D(B) \times D(A)$ such that $y(t) = \psi_0(A,B,t)f_0 + \psi_1(A,B,t)f_1$ $(t \in R_+)$ is a solution of the Cauchy problem for (1) on R_+ with i.d. (f_0, f_1).

Let us assume that (f_0, f_1) is such a vector.

Here $f_0 \in D(B)$. On the other hand, by virtue of Theorem 3.2, for $f_0 \in D(B)$: $y_0(t) = \psi_0(A,B,t)f_0$ $(t \in R_+)$ is a solution of (1) on R_+ such that $y_0(0) = f_0$,

$y_0'(0) = 0$. So, $y_1(t) = y(t) - y_0(t) = \psi_1(A,B,t)f_1$ $(t \in R_+)$ is a solution of (1) on R_+ such that $y_1(0) = 0$, $y_1'(0) = f_1$. Moreover, $f_1 \in D(A)$.

Conversely, if $f_0 \in D(B)$ and $f_1 \in D(A)$ is such that $y_1(t) = \psi_1(A,B,t)f_1$ $(t \in R_+)$ is a solution of (1) on R_+ with $y_1(0) = 0$, $y_1'(0) = f_1$, then $y(t) = \psi_0(A,B,t)f_0 + \psi_1(A,B,t)f_1$ $(t \in R_+)$ is a solution of the Cauchy problem for (1) on R_+ with i.d. (f_0, f_1).

Summarizing, F_C is the set of all $(f_0, f_1) \in D(B) \times D(A)$ such that $y_1(t) = \psi_1(A,B,t)f_1$ $(t \in R_+)$ is a solution of (1) on R_+ with $y_1(0) = 0$ and $y_1'(0) = f_1$.

In view of Definition 1.1, the last condition on $f_1 \in D(A)$ contains the following one: for all $t \in R_+$, $y_1(t) = \psi_1(A,B,t)f_1 \in D(B)$, and $By_1(t) = B\psi_1(A,B,t)f_1$ is continuous in H on R_+. In particular,

$$\| By_1(t) \|^2 = \| B\psi_1(A,B,t)f_1 \|^2$$

is continuous on R_+. In other words,

$$R(t) = \int\limits_{\sigma(A,B)} |\psi_1(\lambda,\mu,t)|^2 |\mu|^2 d(Ef_1, f_1)$$

is finite for all $t \in R_+$ and continuous on R_+.

We arrive at the following problem.

One has a set $P = \sigma(A,B)$ which satisfies (2). Furthermore, one has a σ-finite measure ρ on $P = \sigma(A,B)$ (here $d\rho = |\mu|^2 d(Ef_1, f_1)$). The question is: what conditions must this σ-finite measure ρ on P satisfy for its integral transform $R(t) = \int_P |\psi_1(\lambda,\mu,t)|^2 d\rho_{\lambda,\mu}$ to be finite for all $t \in R_+$ and continuous on R_+?

In Chapter 5, we answer this question in the general case where P is a closed subset of C^2 which satisfies (2) and ρ is a σ-finite measure on P. As well, we answer some other questions related to this one.

In Chapter 6, based on these results, we answer the question 1 on the set of all initial data (f_0, f_1) that ensure the unique solvability of the Cauchy problem for (1) on R_+. Moreover, we answer some other interesting questions related to the question 1.