Progress in Mathematics

Volume 193

Kevin P. Knudson

Homology of Linear Groups

Springer Basel AG

Author:

Kevin P. Knudson
Department of Mathematics
Wayne State University
Detroit, MI 48202
USA
e-mail: knudson@math.wayne.edu

2000 Mathematics Subject Classification 20G10

Knudson, Kevin P. (Patrick), 1969-
Homology of linear groups / Kevin P. Knudson.
p. cm. -- (Progress in mathematics ; v. 193)
Includes bibliographical references and index.
ISBN 978-3-0348-9523-1 ISBN 978-3-0348-8338-2 (eBook)
DOI 10.1007/978-3-0348-8338-2
1. Linear algebraic groups. 2. Homology theory. I. Title. II. Progress in mathematics (Boston, Mass.) ; vol. 193.

QA179 .K59 2000
512'.55--dc21 00-057147

Deutsche Bibliothek Cataloging-in-Publication Data

Knudson, Kevin P.:
Homology of linear groups / Kevin P. Knudson. – Basel ; Boston ; Berlin : Birkhäuser, 2001
(Progress in mathematics ; Vol. 193)
ISBN 978-3-0348-9523-1

Originally published by Birkhäuser Verlag in 2001
Softcover reprint of the hardcover 1st edition 2001
Printed on acid-free paper produced of chlorine-free pulp. TCF ∞

ISBN 978-3-0348-9523-1

9 8 7 6 5 4 3 2 1

To Ellen, for inspiration;
and to Gus, for distraction

Contents

Preface

With his definition of the higher algebraic K-groups of a ring, Daniel Quillen launched a new branch of mathematics. These groups are defined as the homotopy groups of a rather complicated space $BGL(R)^+$. This space is a modification of the classifying space of the infinite general linear group $GL(R) = \bigcup_{n\geq 1} GL_n(R)$; its homology groups coincide with those of the discrete group $GL(R)$. Quillen's calculation of the (co)homology of $GL_n(\mathbb{F}_q)$ and the resulting computation of the K-groups of $\mathbb{F}_q$ were the first results in this new field.

In the intervening 25 years, some of the world's best mathematicians have devoted their energies to the study of Quillen's K-groups. From the very beginning, the importance of the unstable homology groups $H_\bullet(GL_n(R))$ was apparent. Explicit computations have been hard to come by, though. For example, a group as simple as $SL_4(\mathbb{Z})$ has resisted the complete calculation of its homology. Not surprisingly, then, the K-groups of $\mathbb{Z}$ are still mysterious (although a great deal is known; for example, the 2-torsion in $K_\bullet(\mathbb{Z})$ has been determined completely).

This monograph presents the current state of affairs in the study of the homology of linear groups. I have tried to trace the development of the theory chronologically, beginning with Quillen's results and proceeding to the present. This linearity is interrupted occasionally, however.

Chapter 1 is an overview of the early results of the subject. Quillen's calculation of $H^\bullet(GL_n(\mathbb{F}_q))$ is presented along with certain conjectures about the structure of $H^\bullet(GL_n(\Lambda), \mathbb{F}_p)$(due to Quillen and Lichtenbaum), where Λ is a $\mathbb{Z}[1/p]$-algebra. These conjectures have spurred the development of a great deal of interesting mathematics, such as the étale K-theory of W. Dwyer and E. Friedlander. Chapter 1 also includes a discussion of A. Borel's calculation of the stable cohomology of arithmetic groups (e.g., $SL(\mathbb{Z})$) and the resulting consequences for the K-theory of rings of integers in number fields. There is also a brief discussion of congruence subgroups. A thorough discussion of much of the material in Chapter 1 would require an entire book; for this reason, I have chosen simply to outline or omit proofs of several results. The major exception to this is the calculation of $H^\bullet(GL_n(\mathbb{F}_q))$. When proofs are truncated, a reference is always provided.

Chapter 2 presents the known stability results for $H_\bullet(GL_n)$. The basic idea is to study the extent to which the groups $H_i(GL_n)$ stabilize as n increases. We first discuss a theorem of W. van der Kallen, a very general result in this area, and we prove a special case, due to Y. Nesterenko and A. Suslin, for so-called rings with many units. This stability result is then used to glean information about the K-groups of local rings. Chapter 2 also contains some results about the rank conjecture for infinite fields.

Chapter 3 is concerned with low-dimensional homology groups, especially the group $H_3(GL_2)$. This group has a surprising connection to the study of scissors congruence classes of polytopes in hyperbolic space, an idea studied extensively by J. Dupont and C.-H. Sah. The relation between H_3 and the so-called Bloch group (due to Suslin) is also discussed, along with generalizations and extensions due to the author, S. Yagunov, and P. Elbaz–Vincent. A connection between the Bloch group and hyperbolic 3-manifolds, discovered by W. Neumann and J. Yang, is also presented.

Rank one groups are the focus of Chapter 4. It turns out that the homology of SL_2 and PGL_2 over certain rings is computable via actions of these groups on trees. The homology of the following groups is computed completely: $SL_2(\mathbb{Z}[1/p])$ (Adem–Naffah), $SL_2(k[t])$, $SL_2(k[t,t^{-1}])$ (Knudson), $PGL_2(k[C])$ where C is a smooth affine curve of the form $X - \{p\}$, where X is projective over an infinite field k (Suslin, Knudson). The generalization of these results to groups of higher rank (due to Henn, Soulé, and the author) is also discussed, along with various applications.

The final chapter (Chapter 5) provides a comprehensive account of the Friedlander–Milnor conjecture concerning the homology of algebraic groups made discrete. All known cases—solvable Lie groups (Milnor); solvable algebraic groups, tori, normalizers of maximal tori (Jardine); the stable groups GL (Suslin, Jardine); arbitrary groups over $\overline{\mathbb{F}}_p$ (Friedlander–Mislin)—are discussed. The low-dimensional cases H_1, H_2 (Sah, Milnor) and $H_3(GL_2)$ (Sah, Knudson) are also proved.

There are three appendices. The first provides a brief overview of the homology of discrete groups. The second recalls the basic notions of classifying spaces and the definition of K-theory (topological and algebraic). The third covers the fundamentals of étale cohomology.

Also, I have included exercises at the end of each chapter. These vary from the routine to the very challenging. I hope they will prove to be useful to graduate students.

This book should be accessible to graduate students who have a good working knowledge of algebraic topology and the fundamentals of group cohomology. Indeed, many of the calculations presented here are interesting applications of the spectral sequence techniques introduced in Chapter VII of K. Brown's excellent book [**21**]. An acquaintance with the basics of algebraic groups is helpful, but if the reader prefers, the following translation may be used: reductive group: GL_n, semi-simple group: SL_n, Borel subgroup: upper triangular matrices, torus: diagonal subgroup, unipotent group: upper triangular matrices with 1's on the diagonal, Weyl group: the symmetric group on n letters. Spectral sequences are used extensively; the construction of the spectral sequence associated to a filtered complex is reviewed in Appendix A. No extensive knowledge of algebraic geometry or étale cohomology is required; however, we do assume that the reader understands the basics of scheme theory.

I would like to thank all those with whom I have had the opportunity to discuss the material in this book. These include Dick Hain, Andrei Suslin, Eric Friedlander, Rick Jardine, Howard Garland, John Harer, Jun Yang, Marc Levine, Chuck Weibel, Mark Walker, Serge Yagunov, Philippe Elbaz–Vincent, Burt Totaro, Rob de Jeu, and Hans-Werner Henn. I owe a special debt to Professor Suslin for freely sharing his ideas and for being an inspirational influence overall. I am also grateful to a pair of anonymous referees whose comments significantly enhanced the quality of this monograph.

The first draft of this book was written while I was supported by a National Science Foundation Postdoctoral Fellowship at Northwestern University. I thank the NSF and Northwestern for their generous support.

Detroit, Michigan K. K.
April 2000