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# **Infinite Dimensional Kähler Manifolds**

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Editors**

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# Contents

<b>Preface</b> .....	xi
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## **Introduction to Group Actions in Symplectic and Complex Geometry**

*Alan Huckleberry*

<b>I. Finite-dimensional manifolds</b> .....	1
1. Vector space structures .....	1
2. Local theory .....	4
3. Global differentiable objects .....	5
4. A sketch of integration theory .....	13
5. Smooth submanifolds .....	15
6. Induced orientation and Stokes' theorem .....	17
7. Functionals on de Rham cohomology .....	19
<b>II. Elements of Lie groups and their actions</b> .....	20
1. Introduction to actions and quotients .....	20
2. Examples of Lie groups .....	21
3. Smooth actions of Lie groups .....	26
4. Fiber bundles .....	35
<b>III. Manifolds with additional structure</b> .....	40
1. Geometric structures on vector spaces .....	40
2. The elements of function theory .....	46
3. A brief introduction to complex analysis in higher dimensions .....	53
4. Complex manifolds .....	58
5. Symplectic manifolds .....	70
6. Kähler manifolds .....	78
<b>IV. Symplectic manifolds with symmetry</b> .....	82
1. Introduction to the moment map .....	82
2. Central extensions .....	85
3. Existence and uniqueness of the moment map .....	88
4. Basic examples of the moment map .....	89
5. The Poisson structure on $(\text{Lie } G)^*$ and on coadjoint orbits .....	92
6. The basic formula and some consequences .....	94
7. Moment maps associated to representations .....	94

<b>V. Kählerian structures on coadjoint orbits of compact groups and associated representations</b> .....	95
1. Generalities on compact groups .....	95
2. Root decomposition for $\mathfrak{t}^{\mathbb{C}}$ .....	98
3. Complexification of compact groups .....	102
4. Algebraicity properties of complexifications of compact groups .....	105
5. Compact complex homogeneous spaces .....	107
6. The root groups $SL_2(\alpha)$ and $H_2(G/P, \mathbb{Z})$ .....	115
7. Representations of complex semisimple groups .....	123
<b>Literature</b> .....	128

## Infinite-dimensional Groups and their Representations

*Karl-Hermann Neeb*

<b>Introduction</b> .....	131
<b>I. Calculus in locally convex spaces</b> .....	132
1. Differentiable functions .....	133
2. Differentiable functions on Banach spaces .....	135
3. Holomorphic functions .....	138
4. Differentiable manifolds .....	140
5. Infinite-dimensional Lie groups .....	142
<b>II. Dual spaces of locally convex spaces</b> .....	144
1. Metrizable spaces .....	146
2. Semireflexivity .....	148
3. Completeness properties of the dual space .....	150
<b>III. Topologies on function spaces</b> .....	153
1. The space $C^\infty(M, V)$ .....	153
2. Smooth mappings between function spaces .....	158
3. Applications to groups of continuous mappings .....	159
4. Spaces of holomorphic functions .....	160
<b>IV. Representations of infinite-dimensional groups</b> .....	164
<b>V. Generalized coherent state representations</b> .....	168
1. The line bundle over the projective space of a topological vector space .....	168
2. Applications to representation theory .....	172
<b>References</b> .....	177

**Borel-Weil Theory for Loop Groups**

*Karl-Hermann Neeb*

**Introduction** ..... 179

**I. Compact groups** ..... 180

**II. Loop groups and their central extensions** ..... 181

    1. Groups of smooth maps ..... 181

    2. Central extensions of loop groups ..... 184

    3. Appendix IIa: Central extensions and semidirect products ..... 185

    4. Appendix IIb: Smoothness of group actions ..... 188

    5. Appendix IIc: Lifting automorphisms to central extensions ..... 189

    6. Appendix IId: Lifting automorphic group actions  
        to central extensions ..... 192

**III. Root decompositions** ..... 197

    1. The Weyl group ..... 198

    2. Root decomposition of the central extension ..... 200

**IV. Representations of loop groups** ..... 201

    1. Lowest weight vectors and antidominant weights ..... 203

    2. The Casimir operator ..... 206

**V. Representations of involutive semigroups** ..... 209

**VI. Borel-Weil theory** ..... 213

**VII. Consequences for general representations** ..... 226

**References** ..... 228

**Coadjoint Representation of Virasoro-type Lie Algebras  
and Differential Operators on Tensor-densities**

*Valentin Yu. Ovsienko*

**Introduction** ..... 231

**I. Coadjoint representation of Virasoro group and Sturm-Liouville operators;  
Schwarzian derivative as a 1-cocycle** ..... 233

    1. Virasoro group and Virasoro algebra ..... 233

    2. Regularized dual space ..... 234

    3. Coadjoint representation of the Virasoro algebra ..... 235

    4. The coadjoint action of Virasoro group and  
        Schwarzian derivative ..... 236

    5. Space of Sturm-Liouville equations as a  $\text{Diff}^+(S^1)$ -module ..... 236

6. The isomorphism .....	237
7. Vect( $S^1$ )-action on the space of Sturm-Liouville operators .....	238
<b>II. Projectively invariant version of the Gelfand-Fuchs cocycle and of the Schwarzian derivative .....</b>	<b>238</b>
1. Modified Gelfand-Fuchs cocycle .....	239
2. Modified Schwarzian derivative .....	239
3. Energy shift .....	240
4. Projective structures .....	240
<b>III. Kirillov's method of Lie superalgebras .....</b>	<b>241</b>
1. Lie superalgebras .....	241
2. Ramond and Neveu-Schwarz superalgebras .....	242
3. Coadjoint representation .....	243
4. Projective equivariance and Lie superalgebra $osp(1 2)$ .....	243
<b>IV. Invariants of coadjoint representation of the Virasoro group .....</b>	<b>244</b>
1. Monodromy operator as a conjugation class of $\widetilde{SL}(2, \mathbf{R})$ .....	244
2. Classification theorem .....	245
<b>V. Extension of the Lie algebra of first order linear differential operators on <math>S^1</math> and matrix analogue of the Sturm-Liouville operator .....</b>	<b>247</b>
1. Lie algebra of first order differential operators on $S^1$ and its central extensions .....	247
2. Matrix Sturm-Liouville operators .....	247
3. Action of Lie algebra of differential operators .....	248
4. Generalized Neveu-Schwarz superalgebra .....	248
<b>VI. Geometrical definition of the Gelfand-Dickey bracket and the relation to the Moyal-Weil star-product .....</b>	<b>249</b>
1. Moyal-Weyl star-product .....	250
2. Moyal-Weyl star-product on tensor-densities, the transvectants ....	250
3. Space of third order linear differential operators as a $\text{Diff}^+(S^1)$ -module .....	251
4. Second order Lie derivative .....	252
5. Adler-Gelfand-Dickey Poisson structure .....	253
<b>References .....</b>	<b>253</b>

**From Group Actions to Determinant Bundles  
Using (Heat-kernel) Renormalization Techniques**

*Sylvie Paycha*

<b>Introduction</b> .....	257
<b>I. Renormalization techniques</b> .....	260
1. Renormalized limits .....	260
2. Renormalization procedures .....	261
3. Heat-kernel renormalization procedures .....	262
4. Renormalized determinants .....	264
<b>II. The first Chern form on a class of hermitian vector bundles</b> .....	266
1. Renormalization procedures on vector bundles .....	266
2. Weighted first Chern forms on infinite dimensional vector bundles .....	268
<b>III. The geometry of gauge orbits</b> .....	270
1. The finite dimensional setting .....	270
2. The infinite dimensional setting .....	272
<b>IV. The geometry of determinant bundles</b> .....	275
1. Determinant bundles .....	275
2. A metric on the determinant bundle .....	276
3. A connection on the determinant bundle .....	276
4. Curvature on the determinant bundle .....	277
<b>V. An example: the action of diffeomorphisms on complex structures</b> ...	278
1. The orbit picture .....	278
2. Riemannian structures .....	279
3. A super vector bundle arising from the group action .....	280
4. The determinant bundle picture .....	281
5. First Chern form on the vector bundle .....	282
<b>References</b> .....	283

# Fermionic Second Quantization and the Geometry of the Restricted Grassmannian

*Tilmann Wurzbacher*

<b>Introduction</b> .....	287
<b>I. Fermionic second quantization</b> .....	290
1. The Dirac equation and the negative energy problem .....	291
2. Fermionic multiparticle formalism: Fock space and the CAR-algebra .....	293
<b>II. Bogoliubov transformations and the Schwinger term</b> .....	298
1. Implementation of operators on the Fock space .....	298
2. The Schwinger term .....	308
3. The central extensions $U_{\text{res}}^{\sim}$ and $GL_{\text{res}}^{\sim}$ .....	316
<b>III. The restricted Grassmannian of a polarized Hilbert space</b> .....	326
1. The restricted Grassmannian as a homogeneous complex manifold .....	326
2. The basic differential geometry of the restricted Grassmannian .....	332
<b>IV. The non-equivariant moment map of the restricted Grassmannian</b> .....	336
1. Differential $k$ -forms in infinite dimensions .....	336
2. Symplectic manifolds, group actions and the co-moment map .....	343
3. Co-momentum and momentum maps (in infinite dimensions) .....	347
4. Examples of symplectic actions and (co-)momentum maps .....	352
5. The $U_{\text{res}}$ -moment map on $G_{\text{res}}$ and the Schwinger term .....	354
<b>V. The determinant line bundle on the restricted Grassmannian</b> .....	358
1. The $C^*$ -algebraic construction of the determinant bundle DET .....	358
2. Comparison to other approaches to the determinant bundle .....	361
3. Holomorphic sections of the dual of DET .....	366
<b>References</b> .....	370

# Preface

Infinite dimensional manifolds, Lie groups and algebras arise naturally in many areas of mathematics and physics. Having been used mainly as a tool for the study of finite dimensional objects, as for example in four-dimensional gauge theory or in the study of closed geodesics or trajectories of Hamiltonian flows, the emphasis has changed and they are now frequently studied for their own independent interest.

Examples include the representation theory of loop groups and current groups and their Lie algebras, diffeomorphism groups and Lie algebras of vector fields, e.g., the Virasoro algebra, the geometric approach to bosonic and fermionic second quantization, and the analysis and geometry of spaces of paths and loops in finite dimensional Riemannian manifolds. From this point of view, finite dimensional geometry and representation theory becomes more of a helpful guideline rather than the goal of mathematical study of infinite dimensional objects.

Being interested in communicating parts of this highly active subject to advanced students and young researchers, the first named editor and Alexander Kirillov organized a DMV-Seminar on “Infinite dimensional Kähler manifolds” which took place November 19–25, 1995, at the Mathematisches Forschungsinstitut Oberwolfach. Unfortunately, administrative problems prevented Kirillov from attending the seminar. In response, the second named editor joined in the preparations of the DMV-Seminar which finally consisted of a longer introductory course given by Alan Huckleberry and several shorter courses on selected specialized subjects. Let us take the opportunity to mention that, besides those lectures which led to contributions in this volume, there were others which were given by Askar Dzhamal'daev, Peter Heinzner, Patrick Iglesias and the late Giorgio Valli.

We now briefly describe the contents of this volume. At the outset it should be emphasized that, while the basic themes were at least touched upon in the lectures, the contributions here go significantly further. These range from being foundational in nature to expositions which describe recent results which were proved after the time of the seminar. The “Introduction to group actions in symplectic and complex geometry” by Alan Huckleberry is of the former type. Here the theory of differentiable manifolds and group actions is developed almost from scratch. The availability of many good text books on Riemannian geometry and personal taste led the author to emphasize the complex and the symplectic aspects of Kähler geometry, resulting in a concise, though accelerated presentation of the basics of complex and symplectic manifolds as well as Lie group actions on them. As an application a rather detailed account of the geometric realization of the irreducible representations of compact Lie groups, i.e., Borel-Weil theory in finite dimensions, is given. Though principally written as a “crash course”, notably the aforementioned last part can equally well be read as a reminder and a preparation for the Borel-Weil theory for loop groups.

In “Infinite dimensional groups and their representations” Karl-Hermann Neeb extends the basic differential calculus on Fréchet manifolds and Lie groups to the more general setting of manifolds modelled on sequentially complete locally

convex topological vector spaces, in the spirit of, e.g., John Milnor. After recalling some refined material from functional analysis, foundational results on the topology of continuous, smooth and holomorphic functions on infinite dimensional manifolds are proved. This provides both a rigorous and very general framework for the theory of actions and representations of infinite dimensional Lie groups and allows the study of “generalized coherent state representations” beyond the Hilbert space situation.

Neub applies this again in his contribution “Borel-Weil theory for loop groups”, where the theory of irreducible unitary positive energy representations is developed in a geometric manner, i.e., in terms of holomorphic sections of line bundles over Kähler manifolds acted upon transitively by loop groups. Although this approach has been rather well-known since the appearance of the book *Loop groups* by Andrew Pressley and Graeme Segal, the thorough exposition should make this contribution a useful reference. Notably, the unitary structure is obtained quite naturally by use of the theory of positive definite functions.

In “Coadjoint representations of Virasoro-type Lie algebras and differential operators on tensor-densities” Valentin Ovsienko explains a fundamental example of an action in infinite dimensions, namely the coadjoint action of the (Bott-)Virasoro group corresponding to the Virasoro Lie algebra. A complete but accessible account on the relations between this action and numerous important geometric and algebraic concepts, such as the Schwarzian derivative, projective structures on the circle, periodic Sturm-Liouville operators and their monodromy, and Lie superalgebras is given. Along with simple proofs of the central theorems of the subject, in a concluding chapter some natural generalizations of the Virasoro algebra are described.

Although the contribution of Ovsienko is not focussed on it, we would like to recall here the beautiful result that the coadjoint action of the Virasoro group yields invariant Kählerian structures on the Fréchet manifold  $\text{Diff}^+(S^1)/\text{Rot}(S^1)$ , where  $\text{Rot}(S^1)$  denotes the rotation subgroup in the group  $\text{Diff}^+(S^1)$  of all orientation-preserving diffeomorphisms of  $S^1$ .

Sylvie Paycha addresses in “From group actions to determinant bundles using (heat-kernel) renormalization techniques” an important issue of infinite dimensional geometry, namely how to give a sense to quantities that in finite dimensions are defined in terms of traces and determinants. After reviewing and relating different methods of “regularization” or “renormalization” they are applied to two general situations: the problem of characterizing “minimal orbits” in infinite dimensions and the geometry of determinant bundles.

As an example, relevant notably to string theory, the action of the diffeomorphism group of a closed oriented two-dimensional manifold  $Z$  on the infinite dimensional (weakly) Hermitian manifold of metrics of constant curvature equal to  $-1$  on  $Z$  is considered. It is now classical that the  $L^2$ -metric on the space of metrics, or equivalently on the space of almost-complex structures on  $Z$ , induces the (Kählerian) Weil-Petersson metric on Teichmüller space upon going to the quotient space. Here, in Sylvie Paycha’s contribution, the regularized quantities

arising in the orbit picture, respectively, in the determinant bundle picture, are compared.

In “Fermionic second quantization and the geometry of the restricted Grassmannian” Tilmann Wurzbacher gives a detailed account of another important infinite dimensional Kähler manifold:  $G_{\text{res}}$ , the so-called restricted Grassmannian of a polarized Hilbert space. This contribution starts with the Klein-Gordon equation and traces the way to  $G_{\text{res}}$  via the Dirac equation, the negative energy problem and fermionic second quantization, i.e., the representation theory of the CAR-algebra on certain Fock spaces. The homogeneous Kähler geometry of  $G_{\text{res}}$  and the numerous closely related central extensions of infinite dimensional Lie groups and Lie algebras are then studied in detail.

A discussion of symplectic manifolds and group actions in the framework described by Neeb in “Infinite dimensional groups and their representation” allows an explanation of the (non-equivariant) moment map associated to the action of the restricted unitary group on  $G_{\text{res}}$ . Several other examples of infinite dimensional moment maps are sketched.

In the last chapter a  $C^*$ -algebraic geometric approach to the Grassmannian and its determinant bundle is developed. We expect these methods to be very useful for the study of other infinite dimensional Kähler manifolds.

Let us take this opportunity to thank those involved in the organization of the DMV-Seminar, in particular the *Deutsche Mathematiker-Vereinigung* and the *Mathematisches Forschungsinstitut Oberwolfach*. Notably, we would like to thank the director, Matthias Kreck, and the entire staff of the Oberwolfach institute for providing us with optimal conditions during this intense week in November, 1995. Finally, we would like to thank the lecturers whose talks did not lead to a contribution in this volume and all participants for their curiosity and enthusiasm.

Alan Huckleberry and Tilmann Wurzbacher

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