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Infinite Dimensional Kähler Manifolds

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Preface

Infinite dimensional manifolds, Lie groups and algebras arise naturally in many areas of mathematics and physics. Having been used mainly as a tool for the study of finite dimensional objects, as for example in four-dimensional gauge theory or in the study of closed geodesics or trajectories of Hamiltonian flows, the emphasis has changed and they are now frequently studied for their own independent interest.

Examples include the representation theory of loop groups and current groups and their Lie algebras, diffeomorphism groups and Lie algebras of vector fields, e.g., the Virasoro algebra, the geometric approach to bosonic and fermionic second quantization, and the analysis and geometry of spaces of paths and loops in finite dimensional Riemannian manifolds. From this point of view, finite dimensional geometry and representation theory becomes more of a helpful guideline rather than the goal of mathematical study of infinite dimensional objects.

Being interested in communicating parts of this highly active subject to advanced students and young researchers, the first named editor and Alexander Kirillov organized a DMV-Seminar on "Infinite dimensional Kähler manifolds" which took place November 19–25, 1995, at the Mathematisches Forschungsinstitut Oberwolfach. Unfortunately, administrative problems prevented Kirillov from attending the seminar. In response, the second named editor joined in the preparations of the DMV-Seminar which finally consisted of a longer introductory course given by Alan Huckleberry and several shorter courses on selected specialized subjects. Let us take the opportunity to mention that, besides those lectures which led to contributions in this volume, there were others which were given by Askar Dzhumadil'daev, Peter Heinzner, Patrick Iglesias and the late Giorgio Valli.

We now briefly describe the contents of this volume. At the outset it should be emphasized that, while the basic themes were at least touched upon in the lectures, the contributions here go significantly further. These range from being foundational in nature to expositions which describe recent results which were proved after the time of the seminar. The "Introduction to group actions in symplectic and complex geometry" by Alan Huckleberry is of the former type. Here the theory of differentiable manifolds and group actions is developed almost from scratch. The availability of many good text books on Riemannian geometry and personal taste led the author to emphasize the complex and the symplectic aspects of Kähler geometry, resulting in a concise, though accelerated presentation of the basics of complex and symplectic manifolds as well as Lie group actions on them. As an application a rather detailed account of the geometric realization of the irreducible respresentations of compact Lie groups, i.e., Borel-Weil theory in finite dimensions, is given. Though principally written as a "crash course", notably the aforementioned last part can equally well be read as a reminder and a preparation for the Borel-Weil theory for loop groups.

In "Infinite dimensional groups and their representations" Karl-Hermann Neeb extends the basic differential calculus on Fréchet manifolds and Lie groups to the more general setting of manifolds modelled on sequentially complete locally convex topological vector spaces, in the spirit of, e.g., John Milnor. After recalling some refined material from functional analysis, foundational results on the topology of continuous, smooth and holomorphic functions on infinite dimensional manifolds are proved. This provides both a rigorous and very general framework for the theory of actions and representations of infinite dimensional Lie groups and allows the study of "generalized coherent state representations" beyond the Hilbert space situation.

Neeb applies this again in his contribution "Borel-Weil theory for loop groups", where the theory of irreducible unitary positive energy representations is developed in a geometric manner, i.e., in terms of holomorphic sections of line bundles over Kähler manifolds acted upon transitively by loop groups. Although this approach has been rather well-known since the appearance of the book *Loop groups* by Andrew Pressley and Graeme Segal, the thorough exposition should make this contribution a useful reference. Notably, the unitary structure is obtained quite naturally by use of the theory of positive definite functions.

In "Coadjoint respresentations of Virasoro-type Lie algebras and differential operators on tensor-densities" Valentin Ovsienko explains a fundamental example of an action in infinite dimensions, namely the coadjoint action of the (Bott-)Virasoro group corresponding to the Virasoro Lie algebra. A complete but accessible account on the relations between this action and numerous important geometric and algebraic concepts, such as the Schwarzian derivative, projective structures on the circle, periodic Sturm-Liouville operators and their monodromy, and Lie superalgebras is given. Along with simple proofs of the central theorems of the subject, in a concluding chapter some natural generalizations of the Virasoro algebra are described.

Although the contribution of Ovsienko is not focussed on it, we would like to recall here the beautiful result that the coadjoint action of the Virasoro group yields invariant Kählerian structures on the Fréchet manifold $\mathrm{Diff}^+(S^1)/\mathrm{Rot}(S^1)$, where $\mathrm{Rot}(S^1)$ denotes the rotation subgroup in the group $\mathrm{Diff}^+(S^1)$ of all orientation-preserving diffeomorphisms of S^1.

Sylvie Paycha addresses in "From group actions to determinant bundles using (heat-kernel) renormalization techniques" an important issue of infinite dimensional geometry, namely how to give a sense to quantities that in finite dimensions are defined in terms of traces and determinants. After reviewing and relating different methods of "regularization" or "renormalization" they are applied to two general situations: the problem of characterizing "minimal orbits" in infinite dimensions and the geometry of determinant bundles.

As an example, relevant notably to string theory, the action of the diffeomorphism group of a closed oriented two-dimensional manifold Z on the infinite dimensional (weakly) Hermitian manifold of metrics of constant curvature equal to -1 on Z is considered. It is now classical that the L^2-metric on the space of metrics, or equivalently on the space of almost-complex structures on Z, induces the (Kählerian) Weil-Petersson metric on Teichmüller space upon going to the quotient space. Here, in Sylvie Paycha's contribution, the regularized quantities arising in the orbit picture, respectively, in the determinant bundle picture, are compared.

In "Fermionic second quantization and the geometry of the restricted Grassmannian" Tilmann Wurzbacher gives a detailed account of another important infinite dimensional Kähler manifold: G_{res}, the so-called restricted Grassmannian of a polarized Hilbert space. This contribution starts with the Klein-Gordon equation and traces the way to G_{res} via the Dirac equation, the negative energy problem and fermionic second quantization, i.e., the respresentation theory of the CAR-algebra on certain Fock spaces. The homogeneous Kähler geometry of G_{res} and the numerous closely related central extensions of infinite dimensional Lie groups and Lie algebras are then studied in detail.

A discussion of symplectic manifolds and group actions in the framework described by Neeb in "Infinite dimensional groups and their representation" allows an explanation of the (non-equivariant) moment map associated to the action of the restricted unitary group on G_{res}. Several other examples of infinite dimensional moment maps are sketched.

In the last chapter a C^*-algebraic geometric approach to the Grassmannian and its determinant bundle is developed. We expect these methods to be very useful for the study of other infinite dimensional Kähler manifolds.

Let us take this opportunity to thank those involved in the organization of the DMV-Seminar, in particular the *Deutsche Mathematiker-Vereinigung* and the *Mathematisches Forschungsinstitut Oberwolfach.* Notably, we would like to thank the director, Matthias Kreck, and the entire staff of the Oberwolfach institute for providing us with optimal conditions during this intense week in November, 1995. Finally, we would like to thank the lecturers whose talks did not lead to a contribution in this volume and all participants for their curiosity and enthusiasm.

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