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One-dimensional Functional Equations

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Preface

This monograph is devoted to the study of functional equations

$$g(x, \varphi(x), \varphi(F_1(x)), \ \dots\ , \varphi(F_n(x))) \ = \ 0, \qquad x \ \in \ M, \tag{0.1}$$

where M is either the real line $\mathbb{R}$ or the unit circle $\mathbb{T}$, φ is an unknown function, $F_1, \dots, F_n$ are given mappings of the manifold M into itself and $g : M \times \mathbb{R}^{n+1} \to \mathbb{R}$ is a given mapping.

We assume that the mappings $f, F_1, \dots, F_n$ are of a class C^k, $k \in \mathbb{N} \cup \{\infty, \aleph\}$. According to the standard definition $C^0(M)$ is the class of continuous functions on M; $C^k(M)$ with $0 < k < \infty$ consists of k-times continuously differentiable functions on M;

$$C^\infty(M) = \bigcap_{1<k<\infty} C^k(M)$$

and, in addition, $\mathbb{C}^\aleph$ is the class of analytic functions on M.

The main question related to equation (0.1) is whether it is solvable in one of the above classes.

Among the most known particular cases of (0.1) there are the classical *equation on implicit functions*

$$g(x, \varphi(x)) \ = \ 0, \quad x \ \in \ M, \tag{0.2}$$

which does not contain the transformed argument at all, the *Abel equation*

$$\varphi(F(x)) \ - \ \varphi(x) \ = \ 1, \quad x \ \in \ M, \tag{0.3}$$

or the more general *cohomological equation*

$$\varphi(F(x)) \ - \ \varphi(x) \ = \gamma(x), \quad x \ \in \ M, \tag{0.4}$$

and the *Schröder equation*

$$\varphi(F(x)) \ = \ H(\varphi(x)), \quad x \ \in \ M, \tag{0.5}$$

which systematically appears in dynamical problems.

In particular, if $M = \mathbb{R}$ and $F(x) = x+1$, then (0.4) turns out to be a difference equation

$$\varphi(x+1) - \varphi(x) = \gamma(x), \quad x \in \mathbb{R}, \tag{0.6}$$

completely investigated long ago. It is well known that for an arbitrary function γ there exists a solution φ of (0.6) belonging to the same class C^k as γ.

The general solvability problem for equation (0.1) may be divided in two parts: first, to find conditions which guarantee the existence of its local solution in a neighborhood of some point $x_0 \in M$ and, second, granted such local solvability at every point $x \in M$, to find additional conditions for the global solvability on M. Let us illustrate this approach on the equation

$$\varphi(x) = f(x, \varphi(F(x))), \qquad x \in \mathbb{R}, \tag{0.7}$$

in classes C^k, $0 \leq k \leq \infty$.

If a point $x_0 \in \mathbb{R}$ is such that $F(x_0) \neq x_0$, then it is easy to construct a local C^k-solution in a neighborhood V of x_0. Namely, let $\overline{F(V)} \cap \overline{V} = \emptyset$. Then we can choose an arbitrary C^k-function $\varphi_0(x), x \in F(V)$, define $\varphi_1(x) = f(x, \varphi_0(F(x))), x \in V$, and set

$$\varphi(x) = \begin{cases} \varphi_0(x), & x \in F(V), \\ \varphi_1(x), & x \in V. \end{cases}$$

The function φ may be extended to a C^k-function on a connected neighborhood of the set $\overline{V} \cup \overline{F(V)}$. This yields a local solution since (0.7) is fulfilled for $x \in V$.

If x_0 is a fixed point of F and $y_0 = f(x_0, y_0)$, then various fixed point theorems in functional spaces may be applied to construct a local solution.

If the mapping $F(x)$ has no more than one fixed point and the C^k-mapping $G(x,y) = (F(x), f(x,y))$ is C^k-invertible, the above local solutions can be extended step-by-step via the equation itself as C^k-solutions on the whole axis $\mathbb{R}$. This extension process completely solves the problem of the global solvability modulo local solvability.

The approach described above was applied by many authors to the study of rather general functional equations. Numerous results on solvability of such equations in integrable, smooth, analytic, monotonic, convex functions and on the properties of solutions are represented in monographs [53, 54]. Both monographs contain very detailed lists of references. The current state of affairs is described in the survey [12].

Some simple examples show that a local solvability at every point may not imply global solvability. In particular, this is the case when F in (0.7) has more than one fixed point. A similar effect in the theory of differential equations is known as Stocks phenomenon. This is a situation where an equation has local solutions on overlapping domains covering a neighborhood of a point x_0, but there are obstacles to gluing them to a solution in a whole neighborhood.

Similar obstacles arise for functional equations solvable in a neighborhood of every point $x \in M$ and we describe conditions which permit us to glue such local solutions in a global solution on M.

The material of the book is organized as follows.

Chapter 1 is dedicated to standard questions related to the problem of implicit functions. In addition to the classical theorem on implicit functions we consider relations between formal and local solvability and between local and global solvability.

In Chapter 2 we investigate equations (0.3)–(0.5) closely related to the problem of conjugacy and semi-conjugacy of one-dimensional dynamical systems.

Properties of a mapping which are invariant with respect to transformations of variables determine its dynamical behavior. In turn this behavior determines properties of the related functional equations. We study invariants of mappings and find conditions for two given mappings to be conjugate in the corresponding class of smoothness.

In Section 1, Chapter 2 we consider the fixed-point free diffeomorphisms of the real line and prove that they are conjugate with the shift $x \rightarrow x+1$ and that the conjugating transformation may be chosen from the same smoothness class as the diffeomorphisms themselves. While not very complicated for smooth diffeomorphisms, the proof becomes more technically involved for real-analytic diffeomorphisms. For example, one of these proofs appeals to the uniqueness of the smooth structure on the real line, while another is based on the KAM method.

In Section 2, Chapter 2 we investigate equations (0.3) and (0.4) on the real line with arbitrary mappings F, maybe non-invertible. The absence of nonwandering points of F is a necessary condition for the solvability of (0.3) in continuous functions. If F is injective, then this condition is sufficient for the solvability not only of (0.3) but also of (0.4) with arbitrary function $\gamma(x)$. However, if F is not injective, then additional obstacles to the solvability of the Abel equation arise. We introduce a notion of wandering sets for F, and prove that equation (0.4) is solvable in continuous functions for every continuous γ if and only if every compact set is wandering for F.

Section 3, Chapter 2 is devoted to a local classification of mappings in a neighborhood of a fixed point. These investigations were initiated by Poincaré for analytic functions and by Sternberg for smooth functions. It is well known that if a C^k-mapping $F, k \geq 2$, has a hyperbolic fixed point x_0, then it is reducible to a linear form using a local transformation of the same smoothness, including analytic functions. It is known that this statement is not true for class C^1. The results related to the non-hyperbolic case are less known. The first problem arising here is the problem of a formal classification, i.e., the conjugacy in formal power series. We give a complete list of formal normal forms. Further, we describe a new approach to constructing local invariants in a neighborhood of a fixed point of arbitrary type. This approach is based on the conjugacy of F with a standard shift in the left and right semi-neighborhoods of the fixed point. Using a pair of such straightening mappings we construct a local invariant of smooth conjugacy. In this way we state a criterion of C^1-linearization in a neighborhood of a hyperbolic fixed point and construct an infinite family of not smoothly conjugate local diffeomorphisms with a joint fixed hyperbolic point and the same derivative at this point. The same approach permits us to prove that generically the formal C^∞-conjugacy implies the local C^∞-conjugacy. In the analytic case this approach results in the well-known Ecalle–Voronin moduli.

In Section 4, Chapter 2 we study a global classification of one-dimensional diffeomorphisms with isolated fixed points. One of the possible approaches just copies the above scheme using straightening diffeomorphisms on the complement of the set of fixed points and is applicable for fixed points of arbitrary type. If all such points are hyperbolic, then to construct a complete system of invariants a more effective method is used based on the local linearization of diffeomorphisms.

The last section of Chapter 2 is devoted to similar results for one-dimensional flows.

Chapter 3 is dedicated to the study of the equation

$$\varphi(F(x)) = g(x, \varphi(x)) \tag{0.8}$$

whose properties are determined by the dynamical properties of the mapping

$$G(x, y) = (F(x), g(x, y)).$$

In Section 1, Chapter 3 we prove that if $F(x_0) \neq x_0$ and $F'(x) > 0$, then there exists a local transformation

$$\Phi(x, y) = (H(x), T(x, y))$$

which reduces (0.8) to the simplest difference equation

$$\psi(x + 1) = \pm\, \psi(x). \tag{0.9}$$

This allows us not only to prove a local solvability of (0.8) but also to describe the set of all local solutions. A similar problem is studied in a neighborhood of a fixed point, which is traditional in the theory of functional equations started by Poincaré.

In Section 2, Chapter 3 we consider equation (0.8) in the large on $\mathbb{R}$. It turns that if Fix $F = \emptyset$, then a reduction of (0.8) to (0.9) is possible on the entire real line, at least in non-analytic classes.

In Section 3, Chapter 3, linear operators

$$L\varphi(x) = \varphi(F(x)) - b(x)\varphi(x) \tag{0.10}$$

are classified with respect to transformations

$$\Phi(x, y) = (H(x), S(x)y + h(x))$$

which are affine in y and preserve the linearity of (0.8) if the latter was linear. For a classification of linear operators (0.10) we use a "cohomological" approach similar to that used in the problem of conjugacy from Chapter 2. In particular, we state the conditions for solvability of the equation $L\varphi = \gamma$ in cohomological terms applicable both to smooth and analytic functions.

In Chapter 4 we consider equations (0.1) with $n \geq 2$. The properties of these equations are determined by the joint dynamical behavior of the family $\{ F_i \}$. First of all, we introduce the notion of an absorber of the family. The meaning of this notion is clarified by the fact that if $\varphi_0(x)$ is a solution of (0.1) in some neighborhood of an absorber, then under mild conditions of invertibility imposed on $g(x, y_1, \ldots, y_n)$ the function $\varphi_0(x)$ is extendable as a solution of (0.1) for all x. Moreover, every local solution of (0.1) in a neighborhood of the intersection of a system of absorbers is extendable as a global solution. In particular, the solvability holds if the intersection of all absorbers is empty.

The methods of Chapters 3 and 4 are applicable to the equation

$$\varphi(f(x, \varphi(x))) = g(x, \varphi(x))$$

describing the invariant curves in the plane [8, 53], as well as other equations containing superpositions of an unknown function [19].

In Chapter 5 we study the linear operators

$$L\varphi(x) = \sum_{k=1}^{m} a_k(x)\varphi(F_k(x)) \tag{0.11}$$

from the point of view of the general theory of operators in linear topological spaces. We obtain some conditions of surjectivity, of normal solvability and of the Fredholm property for such operators L.

As a matter of fact, many methods used in the present monograph are applicable to and even in some cases first appeared in multi-dimensional situations. Among them there are the gluing of local solutions from Chapters 2 and 3, the KAM method from Chapter 3, the decomposition method from Chapter 4, various theorems on fixed points, and some others.

Let us make several remarks on notation. We use the symbol C^k to denote various classes of smooth or analytic scalar functions defined on some subsets of the real line or circle. If the upper index k is a finite integer, then C^k is the set of all k-times continuously differentiable functions. Class C^∞ is the set of all infinitely differentiable functions. Finally, $C^\aleph$ is the set of all real-analytic functions.

An invertible mapping F will be called the C^k-diffeomorphism if it and its inverse belong to the class C^k. We do not distinguish between C^0-diffeomorphisms and homeomorphisms.

We first became interested in functional equations during our years with the Mathematical Division of the B.Verkin Institute of Low Temperature Physics and Engineering in Kharkov, Ukraine. At the Institute, we enjoyed all advantages of being members of a mathematical community of exceptionally high standards. This interest intensified after we joined the Mathematical Department of the Ben-Gurion University of the Negev, Beer-Sheva, Israel, where many of the results included in the present monograph were obtained with generous grant support from the Israel Science Foundation. Our most sincere acknowledgements go to both of these institutions for their support.

During our student years, we were taught by Professor Yu.I.Lyubich who supervised our work leading to our PhD dissertations. Prof. Lyubich, who organized and chaired many permanent seminars where we presented our first results, eventually became our co-author in many joint publications. We are greatly thankful to him for his involvement in our lives and careers. His influence on us as mathematicians is reflected in this monograph.