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Parabolic Quasilinear Equations Minimizing Linear Growth Functionals

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Ferran Sunyer i Balaguer (1912–1967) was a self-taught Catalan mathematician who, in spite of a serious physical disability, was very active in research in classical mathematical analysis, an area in which he acquired international recognition. His heirs created the Fundació Ferran Sunyer i Balaguer inside the Institut d'Estudis Catalans to honor the memory of Ferran Sunyer i Balaguer and to promote mathematical research.

Each year, the Fundació Ferran Sunyer i Balaguer and the Institut d'Estudis Catalans award an international research prize for a mathematical monograph of expository nature. The prize-winning monographs are published in this series. Details about the prize and the Fundació Ferran Sunyer i Balaguer can be found at

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This book has been awarded the
Ferran Sunyer i Balaguer 2003 prize.

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Alexander Lubotzky and Dan Segal
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Contents

Preface

Our goal in this monograph is to present general existence and uniqueness results for quasilinear parabolic equations whose operator is, in divergence form, the subdifferential of a Lagrangian which is convex in $|\nabla u|$ and has linear growth as $|\nabla u| \to \infty$. We devote particular attention to the case of the minimizing *total variation flow* for which we study the Neumann, Dirichlet and Cauchy problem in $\mathbb{R}^N$ together with the main qualitative properties of its evolution. This kind of problem appears in different contexts: image processing, faceted crystal growth, continuum mechanics, etc. Motivated by the use of the *total variation model in image restoration*, we started our study of the *minimizing total variation* (TV) *flow* in collaboration with C. Ballester, by studying the corresponding Neumann and Dirichlet problems [13], [14]. Later, in a joint paper with J. I. Diaz [15] we studied the asymptotic behaviour of the solutions of these problems. This study was continued in [34] where some extinction profiles were identified. In particular, this provided some explicit solutions of the denoising problem in image processing. The techniques developed for the total variation flow were extended to cover the case of general convex Lagrangians with linear growth rate in the modulus of the gradient, providing a general existence and uniqueness result in this case [16],[17]. Energy functionals with linear growth appear in different contexts, two classical examples being the nonparametric area integrand $f(\xi) = \sqrt{1 + \|\xi\|^2}$, which is associated with the time-dependent minimal surface equation, and the Hencky model in plasticity.

Let us summarize the contents of this book.

Chapter 1 is devoted to the study of the variational approach to image restoration based on total variation minimization subject to the constraints given by the image acquisition model. We review the model initially introduced by L. Rudin, S. Osher and E. Fatemi [175] which had, on one hand, a strong influence in the development of variational models in image denoising and restoration, and, on the other, pioneered the use of the BV model in image processing. The chapter contains the proof of the Chambolle–Lions theorem proving that the constraints can be incorporated by means of a Lagrange multiplier, thus justifying the usual numerical approach to the problem. Then we interpret the corresponding Euler–Lagrange equation in terms of partial differential equations by means of the PDE characterization of the subdifferential of the total variation. This result follows as a consequence of the results in [13] and has been presented in [48]. The approach we present here is a simple and direct approach to the characterization of the subdifferential of positively 1-homogeneous convex functionals of the gradient due to F. Alter in his unpublished work [3]. Then we display a few experiments on image restoration obtained with this model. The chapter also contains a review of the main numerical methods used in the variational approach to image restoration. We apologize in advance for any missing work.

In Chapter 2 we study the Neumann problem for the minimizing total variation flow. First we present the main existence and uniqueness results for this problem, which are essentially taken from [13]. Due to the homogeneity of the operator associated with the problem in L^p for any $p \geq 1$ we prove that the semigroup solutions are strong solutions. This, combined with the regularity results for quasi-minimizers of the perimeter, permits us to prove a regularizing effect on the level lines of the solution, a result which also holds for the solution of the restoration problem. The chapter also contains a proof that solutions of the Neumann problem stabilize as $t \to \infty$ by converging to the mean value of the initial datum.

The Cauchy problem for the total variation flow is studied in Chapter 3. The purpose of this chapter is to prove existence and uniqueness of entropy solutions for initial data in $L^1_{loc}(\mathbb{R}^N)$. This will enable us to study in later chapters the main features of the flow in $\mathbb{R}^N$, thus, dismissing the effect of boundary conditions. First, we study the flow in $L^2(\mathbb{R}^N)$. In Section 2 we prove uniqueness of entropy solutions for initial data in $L^1_{loc}(\mathbb{R}^N)$, using Kruzhkov's method of doubling variables. Then we prove existence for initial data in $L^1_{loc}(\mathbb{R}^N)$. We end up with the study of the time regularity of solutions.

Chapter 4 is devoted to a study of the asymptotic behaviour and qualitative properties of the solutions of the total variation flow in $\mathbb{R}^N$. We start by describing some numerically observed features of the flow, namely that local maxima (resp. minima) immediately decrease (resp. increase) and produce flat zones in the solution. For that we shall need some radially symmetric explicit solutions of the flow. We also note that the length of the level curves of the solutions is a decreasing function of time. Our next purpose will be to describe the extinction profile (the solution has a finite extinction time) of compactly supported solutions. This behaviour is described by a function which is the solution of an eigenvalue problem for the operator $-\mathrm{div}\left(\frac{Du}{|Du|}\right)$. The rest of the chapter is devoted to the study of explicit solutions of this eigenvalue problem in the plane. In the radial case, positive solutions can be fully characterized. Then we look for characteristic functions which are solutions of it. This permits characterization of the bounded sets of finite perimeter $\Omega \subset \mathbb{R}^2$ for which the function $u(t,x) = (1 - \frac{\mathrm{Per}(\Omega)}{|\Omega|}t)^+ \chi_\Omega(x)$ is an entropy solution of the minimizing total variation flow in $\mathbb{R}^2$. As an important by-product of the eigenvalue problem, one can obtain explicit solutions of the Rudin–Osher–Fatemi image denoising model. The results of this chapter have been taken from [13], [15], [34].

Chapter 5 is concerned with the Dirichlet problem for the total variation flow. In this case, the homogeneity of the operator is lost, and the notion of entropy solution in the sense of Kruzhkov is required to obtain a uniqueness result. Existence and time regularity of entropy solutions follow from the usual semigroup theory approach. The techniques introduced in this chapter will be the basis for results in the next two chapters dealing with more general operators. The presentation of this chapter is based on [14].

The next two chapters are devoted to a study of the Dirichlet problem for quasilinear parabolic equations whose operator is, in divergence form, the subdifferential of a Lagrangian which is convex and has linear growth in the magnitude of the gradient. More precisely, we study the Dirichlet problem in a bounded domain Ω with boundary datum $\varphi \in L^1(\partial\Omega)$, for the differential operator $-\mathrm{div}\, \mathbf{a}(x, Du)$, where $\mathbf{a}(x,\xi) = \nabla_\xi f(x,\xi)$, f being a convex function of ξ with linear growth as $\|\xi\| \to \infty$. The regularity assumptions we need to impose on the Lagrangian f exclude the total variation flow, i.e., the case $f(\xi) = \|\xi\|$, which was studied in Chapter 5; but we include many examples relevant in applications, like the nonparametric area integrand and Hencky plasticity. In Chapter 6 we prove existence and uniqueness of strong solutions in $L^2(\Omega)$ using the theory of nonlinear semigroups generated by subdifferential operators. Now, to get the full strength of the abstract result derived from semigroup theory, we need to characterize the subdifferential of the energy functional associated with the problem. In Chapter 7 we prove existence and uniqueness of entropy solutions for data in $L^1(\Omega)$. Existence follows by means of Crandall–Ligget's semigroup generation theorem, while uniqueness is proved using again Kruzhkov's method of doubling variables. The results of these two chapters are essentially taken from [16] and [17], respectively.

The book finishes with three appendices in which we outline some of the main tools used in the above chapters. In the first one (Appendix A) we present without proofs the main results of nonlinear semigroup theory which is the main tool used in this text to prove existence of solutions. Due to the linear growth of the energy functionals associated with the problems studied in this monograph, the natural energy space to study them is the space of functions of bounded variation. In Appendix B we outline some of the main points of the theory of functions of bounded variation used in the previous chapters. Finally, following G. Anzelloti's paper [25], Appendix C is devoted to the main results about pairings between measures and bounded measurable functions, one of the fundamental tools of the text.

It is a pleasure to acknowledge here the debt we owe to our coauthors, namely C. Ballester, G Bellettini, J.I. Diaz and M. Novaga. This monograph could not have been written without their contribution. We would like to thank also F. Alter for permitting us to reproduce his unpublished work [3]. We are also indebted to

B. Rougé and the CNES for stimulating discussions about the restoration problem which gave us a better understanding of it, and for his kind permission to reproduce the images of Chapter 1. We thank M. Bertalmio, A. Solé and B. Rougé for providing us these experiments. Finally we are indebted with L. Rudin from Cognitech Inc. for stimulating us to work on the theoretical analysis of the total variation restoration problem which motivated the subsequent work. Thanks should also be extended to many colleagues with whom we have shared their views on image processing and PDEs, among them we would like to thank Ph. Bénilan, J. Blat, A. Chambolle, P.L. Lions, F. Malgouyres, L. Moisan, S. Moll, J.M. Morel, P. Mulet, S. Osher, G. Sapiro, S. Segura, J. Toledo and J.L. Vázquez.

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