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Domain Decomposition Methods in Optimal Control of Partial Differential Equations

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The cover pictures show a set of three membranes coupled along a flexible joint. The first picture shows the initial configuration while the second picture provides a snapshot of the corresponding vibrations after some seconds. The second picture amply demonstrates the transmission of vibrations across the joint. Transverse motion is thereby converted to in-plane motion and vice versa. The results and methods in this monograph concentrate on the decomposition in space and time of such and much more complex dynamics subject to boundary and distributed controls.

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Preface

This monograph considers problems of optimal control for partial differential equations of elliptic and, more importantly, of hyperbolic types on networked domains. The main goal is to describe, develop and analyze iterative space and time domain decompositions of such problems on the infinite-dimensional level.

While domain decomposition methods have a long history dating back well over one hundred years, it is only during the last decade that they have become a major tool in numerical analysis of partial differential equations. A keyword in this context is parallelism. This development is perhaps best illustrated by the fact that we just encountered the 15th annual conference precisely on this topic. Without attempting to provide a complete list of introductory references let us just mention the monograph by Quarteroni and Valli [91] as a general up-to-date reference on domain decomposition methods for partial differential equations.

The emphasis of this monograph is to put domain decomposition methods in the context of so-called virtual optimal control problems and, more importantly, to treat optimal control problems for partial differential equations and their decompositions by an all-at-once approach. This means that we are mainly interested in decomposition techniques which can be interpreted as virtual optimal control problems and which, together with the real control problem coming from an underlying application, lead to a sequence of individual optimal control problems on the subdomains that are iteratively decoupled across the interfaces. This task can be accomplished in a very general way that applies to elliptic, parabolic, and hyperbolic or Petrowski-type systems.

While this is already a formidable task, we also develop a general theory for systems of partial differential equations on multi-linked or networked domains; see the introductory chapter for extensive motivation. It is obvious that from a physical point of view, any domain decomposition of a multi-link system should respect the structural elements as individuals coupled at multiple interfaces. Thus it is the decoupling of multiple interface conditions or, in other words, a nonoverlapping domain decomposition procedure, that one is seeking. The engineering terminology for this approach is 'substructuring'.

As the overall requirement of decomposing, optimizing and simulating systems described by partial differential equations on networked domains is beyond the scope of any in-depth treatment in a single book, in this monograph we have concentrated on the infinite-dimensional level. However, the philosophy outlined above, and made more transparent in the introductory chapter, recommends this selection. Indeed, once the decomposition into local optimal control problems is established and the convergence of the corresponding scheme guaranteed on the infinite-dimensional level, the implementation in parallel of numerical techniques for standard optimal control problems on standard domains is a separate and independent task. The infinite-dimensional algorithmic procedures can thus be seen as a general framework which is completely independent of any discretization scheme. We therefore develop not only the convergence of those schemes in an infinite-dimensional framework but also prove a posteriori error estimates with respect to the matching conditions along multiple interfaces. After local discretization, and after incorporating discrete a posteriori error estimates with respect to finite-dimensional approximations of the local equations into the former ones, the basis is laid for a parallel and adaptive numerical treatment of optimal control problems involving partial differential equations on networked domains. Moreover, the iterative domain decomposition techniques may be used as preconditioners of global-local type.

In order to stay within a common theoretical framework, in this work we consider optimal control problems for elliptic and hyperbolic problems only. Elliptic problems, apart from their natural physical significance, are also more accessible at the theoretical level and are used for introductory expositions. Moreover, after a semi-discretization in time, hyperbolic systems may be viewed as sequences of elliptic problems. We further limit our considerations to linear problems, since a nonlinear theory of domain decompositions of optimal control problems is not available on a general level at this time. In addition, for the sake of clarity and convenience, we consider mainly control problems without control or state constraints. Let us remark, however, that at the expense of some additional technical complexities, one may develop domain decomposition algorithms for all of the examples considered in this book when control constraints are present. Moreover, in Chapter 8 we exhibit a domain decomposition procedure for a state constrained optimal control problem arising from a problem of exact controllability.

Another selection we have made is to consider, in the case of hyperbolic systems, only final value optimal control problems, that is, problems in which the goal is optimal steering of a trajectory from a given initial state to a neighborhood of a specified target state at a specified time. This choice is motivated not only by important physical and engineering applications, but also by the fact that domain decomposition methods for such problems are technically more challenging and far less studied than, for example, domain decomposition methods for trajectory tracking problems, which have been considered extensively in the literature. For exactly the same reasons, we generally restrict our attention to boundary controls as opposed to distributed, or locally distributed, controls; the latter are easily handled by the same theoretical framework that we develop for boundary control problems, and with fewer technical complexities.

Although, as stated, we do not concern ourselves with the numerical implementation of the algorithms that we develop at the infinite-dimensional level, some selected numerical tests are displayed in order to illustrate the efficacy of the methods.

Outline of the Book

In Chapter 1 we give a somewhat detailed introduction to the field of domain decomposition, the state of the art and the scope of this monograph.

In Chapter 2 we give an introduction to some basic ideas typically used in the field of domain decomposition. The material presented is highly selective and is not intended to give a survey of all general concepts of domain decomposition. Rather we have selected those methods and ideas which admit an interpretation as optimal control problems in a wide sense.

We will deal almost exclusively with problems on an infinite-dimensional level. The first part of Chapter 2 is devoted to dynamical problems on one-dimensional domains. We discuss the wave equation and the diffusion-advection equation in an unbounded domain as exemplary models, and describe the most basic decomposition method going back to Schwarz [95] for elliptic problems with overlapping domains. Even on this simple level, the analysis clearly reveals basic features (see [27] and also [24], [25], [26]). For instance, for the wave equation one may develop optimal interface conditions such that the iterative domain decomposition procedure terminates after a few steps. Those interface conditions correspond to transparent boundary conditions which admit complete transmission of data across the interfaces, with no reflections taking place. Typically, however, this desirable property cannot be realized via local-in-time boundary operators across the interface. Similarly, optimal but nonlocal interface operators may be constructed for the diffusion-advection equation such that the iterative domain decomposition procedure terminates after two steps. Also discussed are finite termination and convergence properties for overlapping and nonoverlapping decompositions of bounded one-dimensional domains. It turns out that for the one-dimensional wave equation with transparent boundary conditions at the outer boundary, it is possible to achieve finite termination with local operators. As is to be expected, such is not the case for diffusion-advection equations. A consequence of this discussion is that a general concept of how to construct domain decomposition procedures for nonoverlapping domains begins to appear.

In Section 2.3 some basic concepts for elliptic domain decomposition methods are reviewed. The reader is referred to [91] for an excellent textbook on this material, also with respect to numerical realizations. The classical Dirichlet-Dirichlet method and a Neumann-Neumann method are considered. In both cases the desired transmission condition across the interface can be rewritten as a Steklov-Poincaré type equation. These variants of the Steklov-Poincaré equation may also be interpreted as a controllability constraints. This point of view is not typical in the domain decomposition literature but has been promoted by J.-L. Lions and by the authors of this monograph. Once this interpretation is accepted, many methods which have been developed in the area of optimal control of partial differential equations can be applied.

Subsections 2.3.3–2.3.7 provide a detailed analysis of the most basic algorithm given by P.-L. Lions. The approach that we follow there will be repeated frequently in this book: we first describe the method, then give an easy to access, two domain proof of convergence for the unrelaxed iterations. We then generalize to multi-domain splittings and present interpretations as standard iterative schemes, such as a Richardson iteration, and give a posteriori error estimates. A more explicit treatment of multi-link serial decompositions is then given and the iteration discussed in the context of preconditioned iterations.

Chapter 3 is devoted to partial differential equations on graphs. The concept of linear differential operators of a general vector-valued Sturm-Liouville type on graphs is introduced, after which domain decomposition procedures, which are motivated by the methods and ideas of Chapter 2, are derived. Convergence of the iterations, which are again seen to be canonical Richardson-type iterations, is then established. The last section of Chapter 3 is devoted to hyperbolic problems on graphs, corresponding domain decomposition methods, and their convergence properties.

While Chapters 2 and 3 focus on domain decomposition methods without 'real' controls, Chapter 4 initiates the discussion of domain decomposition methods for optimal control problems for partial differential equations. In order to fix the ideas, the relatively simple, but nonetheless fundamental, elliptic case is treated. First distributed controls, and then boundary controls, are considered. It is shown that the distributed case can be handled by direct application of the methods developed for elliptic problems without control. However, this simple connection ceases in the case of boundary controls. Nevertheless, the basic methods can be extended to this situation as well. After a discussion of the convergence of the iterates we provide, for the first time, a posteriori error estimates for the procedure of domain decomposition of an optimal control problem. The derivation is strongly motivated by the work of Otto and Lube [90].

Chapter 5 is devoted to domain decomposition of optimal control problems for partial differential equations on graphs. Once again we start with the basic elliptic case and discuss distributed as well as boundary control problems. Domain decomposition procedures are formulated and convergence of the corresponding iterative schemes is established. We then proceed to develop and analyze a domain decomposition procedure for optimal final value control problems for linear hyperbolic equations on graphs.

Chapters 6 through 9 are devoted to the study of domain decomposition of optimal control problems for (primarily) hyperbolic partial differential equations in space dimension greater than one. This enterprise starts with Chapter 6, which deals with the domain decomposition of optimal final value control problems for general wave equations with absorbing (dissipative) boundary conditions, this choice being motivated by the spatial domain decompositions in Chapter 2. First of all the model problem is introduced and well-posedness questions are considered in detail. In particular, time and space regularity results are developed. The section on well-posedness is followed by a description of a general time domain decomposition procedure which is reminiscent of those devised by J.-L. Lions et al. [75]. The method is also motivated by the spatial domain decomposition method of

Chapters 2 to 5. The time domain decomposition procedure is formulated, and its convergence is proved. Then pointwise-in-time and uniform a posteriori error estimates are developed. Following that analysis, decomposition of the spatial domain is considered. Once again, convergence of the iterative procedure, and a posteriori error estimates, are provided. The last section of Chapter 6 concentrates then on the possible combinations of time and space domain decomposition procedures.

Chapter 7 is devoted to domain decomposition of optimal final value control problems for the dynamic Maxwell system with a dissipative boundary condition known as the Silver-Müller condition, which is in fact the first-order local approximation to a transparent boundary condition. The flow of the presentation is the same as in Chapter 6 and results analogous to those established there are obtained.

In Chapter 8 we reconsider general wave equations, but this time with a conservative boundary condition. The lack of dissipativity in the boundary condition causes a loss of regularity so that, in particular, when considering optimal final value control the deviation of the final state from the target state must be penalized in a weaker norm than previously. This leads to an optimality system that differs from that occurring when dissipativity is present. While this has little effect on the analysis of our time domain decomposition algorithm, it necessitates significant changes in the spatial domain decomposition algorithm and its analysis. Once again, appropriate convergence results, and a posteriori error estimates, are established.

In addition, in Chapter 8 is presented a spatial domain decomposition procedure for the optimality system associated with the problem of *exact controllability*, that is, the problem of minimum norm control of a trajectory from a given initial state to a prescribed target state. It is shown, in particular, that an appropriate domain decomposition for this optimality system is obtained in the limit, as the penalty parameter goes to infinity, of the domain decomposition algorithm associated with optimal final value control to the target state, discussed earlier in the chapter.

Chapter 9 is concerned with domain decomposition of elliptic equations (without control), and hyperbolic equations (with control), on 2-dimensional polygonal networks in n-dimensional space. The discussion parallels that of Chapters 3 and 5 dealing with such systems on graphs, but with an unavoidable increase in technical complexity.

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