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R. R. Akhmerov
M. I. Kamenskii
A. S. Potapov
A. E. Rodkina
B. N. Sadovskii

Measures of Noncompactness and Condensing Operators

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Authors' addresses:

R.R. Akhmerov
Inst. Comput. Technologies
Lavrentjeva 6
630090 Novosibirsk
USSR

M.I. Kamenskii
B.N. Sadovskii
Voronezh State University
Department of Mathematics
Universitetskaja pl. 1
394693 Voronezh
USSR

A.S. Potapov
Voronezh State Teach. Training Institute
Faculty of Physics and Mathematics
ul. Lenina 86
396611 Voronezh
USSR

A.E. Rodkina
Voronezh Institute of
Civil Engineering
ul. 20 let Oktjabrija 64
394006 Voronezh
USSR

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INTRODUCTION

A condensing (or densifying) operator is a mapping under which the image of any set is in a certain sense more compact than the set itself. The degree of noncompactness of a set is measured by means of functions called measures of noncompactness.

The contractive maps and the compact maps [i.e., in this Introduction, the maps that send any bounded set into a relatively compact one; in the main text the term "compact" will be reserved for the operators that, in addition to having this property, are continuous, i.e., in the authors' terminology, for the completely continuous operators] are condensing. For contractive maps one can take as measure of noncompactness the diameter of a set, while for compact maps can take the indicator function of a family of non-relatively compact sets. The operators of the form $F(x) = G(x, x)$, where G is contractive in the first argument and compact in the second, are also condensing with respect to some natural measures of noncompactness. The linear condensing operators are characterized by the fact that almost all of their spectrum is included in a disc of radius smaller than one.

The examples given above show that condensing operators are a sufficiently typical phenomenon in various applications of functional analysis, for example, in the theory of differential and integral equations.

As is turns out, the condensing operators have properties similar to the compact ones. In particular, the theory of rotation of completely continuous vector fields, the Schauder-Tikhonov fixed point principle, and the Fredholm-Riesz-Schauder theory of linear equations with compact operators admit natural generalizations to condensing operators. Therefore, establishing that a given problem for a differential or integral equation reduces to an equation with a condensing operator yields a considerable amount of information on the properties of its solutions.

The first to consider a quantitative characteristic $\alpha(A)$ measuring the degree of noncompactness of a subset A in a metric space was K. Kuratowski in 1930, in connection with problems of general topology. In the mid Fifties in the works of G. Darbo, L. S. Gol′denshteĭn, I. Gohberg, A. S. Markus, W. V. Petryshyn, A. Furi, A. Vignoli, J. Daneš, Yu. G. Borisovich, Yu. I. Sapronov, M. A. Krasnosel′skiĭ, P. P. Zabreĭko and others various measures of noncompactness were applied in the fixed-point theory, the theory of linear operators, and the theory of differential and integral equations.

This book gives a systematic exposition of the notions and facts connected with measures of noncompactness and condensing operators. The main results are the characterization of linear condensing operators in spectral terms and theorems on perturbations of the spectrum (Chapter 2), and the theory of the index of fixed points of nonlinear condensing operators, together with the ensuing fixed-point theorems (Chapter 3). Chapter 1 is devoted to the main definitions, examples, and simplest properties of measures of noncompactness and condensing operators. In Chapter 4 we consider examples of applications of the techniques developed here to problems for differential equations in Banach spaces, stochastic differential equations with delay, functional-differential equations of neutral type, and integral equations.

In the treatment of the theory itself as well as of its applications we aimed at considering the simplest situation, leaving the comments concerning possible generalizations for the concluding sections or subsections. For additional information the reader is referred to the surveys [10, 28, 160].

The authors use this opportunity to express their gratitude to Mark Aleksandrovich Krasnosel'skiĭ, under whose influence many of the problems discussed here were posed and solved.