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Dispersive Equations and Nonlinear Waves

Generalized Korteweg–de Vries, Nonlinear Schrödinger, Wave and Schrödinger Maps

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Preface

Nonlinear wave equations are ubiquitous in physics and applied sciences; for example, they appear as effective equations in general relativity and elasticity, for water waves, nonlinear optics and superconductivity, in gauge theory and the motion of Bose-Einstein condensates. Despite encompassing a large class of equations, there are recurrent themes: dispersion, solitons and their stability, blow up and scattering. In these notes we want to pursue different coordinated threads: the nonlinear Schrödinger equation and generalized KdV, critical wave and Schrödinger equations, and geometric dispersive equations. Our goal is to introduce the equations and to describe an array of ideas and techniques used in their study, leading up to the current results and remaining open problems.

The first part is devoted to the nonlinear Schrödinger equation, the generalized Korteweg-de Vries equation, and the Kadomtsev-Petviashvili II equation. It introduces basic techniques, stationary phase and Strichartz estimates, the space of functions of bounded p-variation and their adaptation to dispersive equations, convolutions and bilinear estimates. The nonlinear Schrödinger equation and the (generalized) Korteweg-de Vries equation exhibit a fascinating and rich structure. They provide the simplest but nevertheless nontrivial context for many important techniques, as well as the simplest framework for open challenging questions. The last section of the first part describes a scheme for constructing solutions to dispersive equations, often in scale-invariant function spaces; this is demonstrated in the context of the generalized Korteweg-de Vries equation and the Kadomtsev-Petviashvili II equation.

Over the last decade, the induction on energy paradigm has grown into a powerful tool for the large-data analysis of evolution equations. It continues to develop in both depth and breadth, and has already proven useful over a wide range of equations, from semilinear wave and Schrödinger equations to fluid equations and geometric flows. While enjoying many parallels to the calculus of variations and often using its terminology, this new approach requires an independent set of techniques. In these notes we use the energy-critical nonlinear Schrödinger equation as a model to demonstrate these methods and their application to the question of large-data global well-posedness.

Within the field of nonlinear dispersive equations, a special role is played by the so-called geometric dispersive equations, which arise from the standard

Lagrangian or Hamiltonian formalism, but applied in a geometric context. The two simplest examples of such equations are the wave map and Schrödinger map equations. These are discussed in the third part. The emphasis is on wave maps, where even the small-data problem poses new challenges, both of technical nature (function spaces) and conceptually (renormalization). In addition, the large-data problem brings back techniques such as induction on energy and Morawetz estimates. All of this happens on top of a differential geometry layer which needs to be understood first. The corresponding elliptic and parabolic analogues, namely, harmonic maps and the harmonic map heat flow, also play a role. The last section concludes with a discussion of the small-data problem for Schrödinger maps; there the large-data problem is still open.

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