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Special Functions of Mathematical (Geo-)Physics

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Preface

An essential aim of geomathematics is the investigation of qualitative and quantitative structures of the Earth's system to deepen our understanding of its complexity. In this respect, special functions comprise the essential instruments for mathematical interaction of abstraction and concretization. Special functions enable the formulation of a geoscientific problem by reduction such that a new, more concrete problem can be attacked within a well-structured framework, usually in the context of differential equations. A good understanding of special functions provides the capacity to recognize causality between the abstractness of the geomathematical concept and the impact on, as well as cross-sectional importance to, the geoscientific reality.

Our purpose in this work is to present a textbook that allows the reader to concentrate on special fields such as the geosphere, hydrosphere, or atmosphere. In other words, the special functions to be discussed vary widely, depending on the chosen measurement parameters (gravitation, electric and magnetic fields, deformation, climate observables, fluid flow, etc.) and on the field characteristic (potential field, diffusion field, wave field). The differential equation under consideration determines the type of special functions that are needed in the desired reduction process.

The diversity of geomathematical problems involves such a large number of scientific manifestations that our approach to any of them has to be selective. In consequence, since greater weight has to be given to some topics than to others, we have chosen to restrict ourselves to gravitation, geomagnetism, elasticity, and fluid flow theory. Gravitational field theory defines a canonical need to generate special function systems for the Laplace equation. Geomagnetism and electric current systems are closely related to the (pre-)Maxwell equation; the deformation of the solid Earth leads to function systems solving the Cauchy–Navier equation (at least when linear material behavior is assumed). Oceanic circulation and wind motion have to be handled in terms of vectorial function systems involving the Navier–Stokes equation or modifications of it. Unfortunately, we are confronted with the difficult challenge to characterize special function systems under adequate consistency in terms of less mathematically structured geometric features of a reference model (such as the geoid or the real Earth's surface) as well as the intrinsic structure of underlying differential equations involving the laws of physics. Thus, at the present stage of geoscience, no compendium can be expected that is both geometrically consistent with modern navigation results and geophysically reflected by advanced mathematical settings. The complexity of a real "potato-like" Earth model is a striking obstacle that can only be overcome to some extent in today's mathematics. Accordingly, the principles lie in the suitable transition to a regularly structured geometry for the Earth, namely, the ball in first approximation. This leads us to a prestructured framework, namely, spherically oriented special function systems.

Looking at the special functions available in the geophysical literature today, we find that a spherical shape of the Earth is used in almost all publications. Indeed, by modern satellite positioning methods, the maximum deviation of the actual Earth's surface from the average Earth's radius (6,371 km) can be determined to be less than 0.4 %. Although a *spheriodization*, i.e., a mathematical formulation simply in spherical reference geometry, amounts to a strong restriction, it is at least acceptable for a large number of problems. Standard special functions since the time of C.F. Gauß are polynomial trial functions, conventionally called spherical harmonics. Spherical harmonics represent the analogs of trigonometric functions for orthogonal (Fourier) expansions on the sphere. In consequence, the use of spherical harmonics in diverse areas of geosciences is a well-established method, particularly for the purpose of decomposing scalar potentials. Nowadays, reference models for the Earth's gravitational and magnetic potential, e.g., are widely known by tables of expansion coefficients of the frequency constituents of their potentials. However, it should be mentioned that vectorial potentials—even in a spherical Earth's reference model—have their own nature. Concerning the mathematical modeling of vector fields, one is usually not interested in their separation into scalar Cartesian component functions. Instead, inherent physical properties should be observed. For example, the external gravitational field is curl-free, the magnetic field is divergence-free, the equations for incompressible flow, i.e., the Navier–Stokes equations, imply divergence-free vector solutions. In a spherical nomenclature as intended in our approach, all these physical constraints result in a formulation by certain operators, such as the surface gradient, surface curl gradient, surface divergence, surface curl. Our types of vector spherical harmonics satisfy these requirements by splitting the tangential part into a curl-free and a divergence-free field, thereby avoiding artificial singularities arising from the use of local coordinates. Basically, two transitions are undertaken in our approach to harmonics: first, the extension from the scalar to the vectorial case is strictly realized under physical constraints and, second, the definition of Legendre functions is canonically described under the phenomenon of rotational invariance on the sphere. The Legendre functions act as constituting elements for zonal functions, i.e., one-dimensional functions only depending on the polar distance of their two arguments. Altogether, the concept of spherical harmonics plays the central role in a geomathematical presentation of special functions, reflecting the significance of a polynomial nature in a spherically shaped Earth. In addition, spherical harmonics comprise the canonical candidates to represent the angular part in a radial/angular decomposition of solution systems for Laplace, Helmholtz, Cauchy–Navier, (pre-)Maxwell, and Navier–Stokes equations.

It is surprising that, besides the geometrically implied spheriodization, the methodologically oriented *periodization* should take some space in a modern collection of special function systems of geomathematical importance. The reasons are twofold. First, the periodization leads back to the Fourier transform in Euclidean spaces that has been well understood for a long time and is extremely efficient in numerical computation. Second, the procedure of periodization leads to the Euler summation formula and the Poisson summation formula which show a close relationship to each other. The Green (lattice) functions forming the essential basis of these summation formulas indeed enable us to express key volume integrals in geophysics, such as the Newton integral, Mie potentials, elastic potentials, by mass lattice point conglomerates that discretely fill out the integration domain under consideration in an equidistributed way.

A variety of examples for *combined periodization and spheriodization* occur in the theory of Earth-satellite relations (cf., e.g., Kaula 1966), mixing time-wise obligations on periodic orbits with space-wise approaches on torus and/or sphere. Satellite gravimetry (see, e.g., Pail and Plank (2002), Sneeuw (2000), Xu et al. (2008), and the references therein) is a particularly interesting area of spaceborne technology, where one-dimensional periodization in time is adequately involved in three-dimensional periodization and/or spheriodization in space.

This textbook presents material used by the Geomathematics Group, University of Kaiserslautern, during the last several years to set up a contemporary theory of special functions of mathematical (geo-)physics. Our work canonically shows a threefold subdivision. Part I provides preparatory material concerning auxiliary functions such as the Gamma function and important classes of orthogonal polynomials. The general concept of orthogonal polynomials is introduced before we start to consider the classical polynomials, in particular the Jacobi polynomials and—as a special and very important case of them—the ultraspherical or Gegenbauer polynomials. Several basic mathematical and physical applications are included, such as quadrature rules, modeling of the electrostatic potential, and the quantum-mechanical description of oscillations. Part II deals with spherically structured function systems. It starts with the scalar theory of spherical harmonics in the Euclidean space $\mathbb{R}^3$ including the addition theorem, the Funk–Hecke formula, as well as the closure and completeness of spherical harmonics in the space of square-integrable functions, i.e., the space of functions with finite signal energy. It follows the physically based theory of vector spherical harmonics. The basic tool to establish divergence-free and curl-free tangential fields is the Helmholtz decomposition theorem. An alternative system of vector spherical harmonics is also constructed in such a way that they can be identified as eigenfunctions of the Beltrami operator. This eigenfunction system plays a particular role in geomagnetism to separate, e.g., the crustal field from other magnetic sources. Both vector spherical harmonic systems are shown to be closed and complete in the space of square-integrable vector fields on the sphere. All properties characterize vector spherical harmonics as suitable trial functions to constitute the angular ingredients in a radial/angular decomposition of solutions of the Cauchy–Navier as well as the Navier–Stokes equation. Part III is devoted to the lattice function as the multi-dimensional,

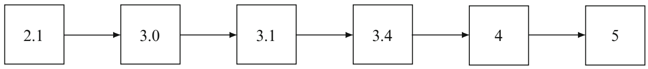

Fig. 1 Selected sections and chapters for a basic one-term course (note that some parts of Sect. 3.3 have to be included to complete Sect. 3.4. Sections 5.5, 5.6, 5.9, and 5.10 can be skipped in the one-term course)

periodic analog to the well-known Bernoulli function. From a physical point of view, the lattice function is interpreted to be the Green function for the Laplace operator corresponding to the boundary conditions of periodicity. It turns out to be the most essential tool for the process of periodization in the context of Euler and Poisson summation formulas. Lattice point sums such as the Zeta and Theta functions, generated by the interaction of point potentials to each other, conclude our multi-periodic theory. It should be remarked that the whole palette of multi-periodic functions is provided in relation to the Laplace operator and arbitrary lattices so that this approach serves as a prototype for further formulations of more general (elliptic) partial differential equations.

Essential ingredients of the textbook are the work of Müller (1952, 1969, 1998), Freeden et al. (1998), Freeden and Schreiner (2009), and Freeden (2011).

Each chapter of the book is followed by exercises related to the presented material. The exercises reflect significant topics, mostly in computational geo-applications. In doing so, they not only confront the reader directly with the contents of the chapter, but also with additional knowledge in geomathematical fields of research, where special functions play a decisive role in applications. Students who wish to continue further studies should consult the literature given as supplements for each topic worked out by exercises. All in all, the content of the book is equally suitable for an education in geomathematics and a study in applied and harmonic analysis.

The book is primarily meant to be a self-consistent introductory text for an advanced undergraduate or graduate course in special functions. The schedule of topics allows a selected subdivision into a one-term course (see Fig. 1) as well as a two-term course. In addition to the proposed sections and chapters in Fig. 1, further contents can be selected from Chap. 3, such as Sect. 3.2 and all details of Sects. 3.3 or 3.5–3.7, if the schedule allows it. The examples of Chap. 1 can be presented at any appropriate time.

A two-term course with special emphasis on particular research fields should include additional material from Chaps. 5 and 7 documenting the special interest of a graduate student in gravitation, geomagnetism, deformation, atmospheric/oceanic flow, respectively. Chapters 6 and 8 give multi-dimensional radial/angular decompositions of harmonic and metaharmonic functions as a reference tool, thereby assuming as preparatory material the whole theory of the Gamma function as presented in Chap. 2. Another separate route going exclusively into the field of lattice functions includes Chaps. 9 and 10 while also requiring all the material of Chap. 2.

In a book of this type, special precautions have been taken to ensure the accuracy of formulas and examples. It is a pleasure to acknowledge with thanks the valuable reading of the manuscript by Dipl.-Math. C. Blick, Dr. C. Gerhards, and Dr. I. Ostermann. We thank our student cand.-phys. Hanna Haug for pointing out some errors and slips in an early version of the manuscript.

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