Birkhäuser

Progress in Mathematics
Volume 303

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Complex Kleinian Groups

Birkhäuser

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ISBN 978-3-0348-0480-6 ISBN 978-3-0348-0481-3 (eBook)
DOI 10.1007/978-3-0348-0481-3
Springer Basel Heidelberg New York Dordrecht London

Library of Congress Control Number: 2012951146

Mathematics Subject Classification (2010): 37F, 22E40, 32M05, 32Q45

Printed on acid-free paper

Springer Basel is part of Springer Science+Business Media (www.springer.com)

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Preface

The purpose of this monograph is to lay down the foundations of the theory of complex Kleinian groups, a concept and a name introduced by José Seade and Alberto Verjovsky in the late 1990s, though their origin traces back to classical work by Henri Poincaré, Emile Picard, Georges Giraud and many others. This brings together several important areas of mathematics, as for instance classical Kleinian group actions, complex hyperbolic geometry, chrystallographic groups and the uniformization problem for complex manifolds. Each of these is in itself a fascinating area of mathematics, with a vast literature, both classical and modern. In fact, real and complex hyperbolic geometry are indeed at the very roots of the theory of complex Kleinian groups and therefore we have devoted the first two chapters of this work to giving a fast overview of these rich areas of mathematics.

A classical Kleinian group is a discrete group of conformal automorphisms of the Riemann sphere $\mathbb{S}^2$, acting on the sphere with a nonempty region of discontinuity. Since the Riemann sphere is biholomorphic to the complex projective line $\mathbb{P}^1_{\mathbb{C}}$, and the orientation preserving conformal automorphisms of $\mathbb{S}^2$ are exactly the elements in $\mathrm{PSL}(2,\mathbb{C})$, one has that in this dimension, the classical Kleinian groups can be regarded too as being groups of holomorphic automorphisms of $\mathbb{P}^1_{\mathbb{C}}$.

When going into higher dimensions, there is a dichotomy: Should we look at conformal automorphisms of the n-sphere $\mathbb{S}^n$?, or should we look at holomorphic automorphisms of the complex projective space $\mathbb{P}^n_{\mathbb{C}}$? These two theories are different because in higher dimensions, neither are conformal maps always holomorphic, nor are holomorphic maps necessarily conformal. In the first case we are talking about groups of isometries of real hyperbolic spaces, an area of mathematics where there is a rich body of knowledge thanks to the contributions of people like Ahlfors, Thurston, Margulis, Sullivan, Mostow, Kapovich, McMullen and many others. In the second case we are talking about an area of mathematics that still is in its childhood, and its study is the theme of this work. Complex Kleinian groups are discrete subgroups of $\mathrm{PSL}(n+1,\mathbb{C})$, the group of holomorphic automorphisms of $\mathbb{P}^n_{\mathbb{C}}$, having a nonempty invariant set where the action is properly discontinuous.

The group $\mathrm{PU}(n,1)$ of holomorphic isometries of complex hyperbolic n-space consists of the elements in $\mathrm{PSL}(n+1,\mathbb{C})$ that preserve a ball, while the affine group $\mathrm{Aff}\,(\mathbb{C}^n) \cong \mathrm{GL}\,(n,\mathbb{C}) \ltimes \mathbb{C}^n$ consists of the elements in $\mathrm{PSL}(n+1,\mathbb{C})$ that leave invariant a given projective hyperplane. These are two very important subgroups of $\mathrm{PSL}(n+1,\mathbb{C})$, but there are others, as for instance all the Lorentz groups $\mathrm{PU}(p,q)$ with $p+q=n+1$, the groups coming via twistor theory, Schottky type groups, and many others. Thus we see that the study of complex Kleinian groups is at the very heart of complex geometry.

An important difference with the classical case springs from the notion of the limit set. We know that if we consider a Kleinian group G acting on the n-sphere $\mathbb{S}^n$, then its limit set Λ is the set of accumulation points of the orbits. Its complement Ω is the region of discontinuity; this is the maximal region where the action is properly discontinuous, and it is also the equicontinuity set of the family of transformations defined by the group action. The same definitions and properties apply to discrete subgroups of $\mathrm{PU}(n,1)$, essentially because when one looks at the action of isometry groups in real or complex hyperbolic space, one has the convergence property, in Misha Kapovich's language. Yet, even in the case of subgroups of $\mathrm{PU}(n,1)$, when we look at the action on the whole space $\mathbb{P}^n_{\mathbb{C}}$ and not only on the unit ball, there is not a well-defined notion of the limit set. There are actually several possible definitions of the limit set, each with its own properties and characteristics, and their study is one of the main features of this monograph.

Another important point to notice is that, in complex dimension 1, we have Sullivan's dictionary (highly enriched by McMullen and others) between the theory of Kleinian groups and the study of iterates of holomorphic functions of the Riemann sphere. There is currently much interesting work being done on iteration theory in several complex variables, and it is to be expected that there should be plenty of analogies (or perhaps a dictionary, to some extent) between iteration theory of endomorphisms of $\mathbb{P}^n_{\mathbb{C}}$ and the theory of complex Kleinian groups. Recent work by W. Barrera, A. Cano, J.- P. Navarrete and others, points in this direction, but there is still a lot to be understood.

We finish this preface by saying that the theory of complex Kleinian groups is a rich area of mathematics that is waiting to be explored. We believe that anyone who absorbs the material in this book, will get plenty of ideas and insights about interesting questions and further lines of research.

Angel Cano,
Juan Pablo Navarrete, and
José Seade.
Cuernavaca and Mérida, México, Spring of 2012.

Contents

Introduction

Kleinian groups were introduced by Henri Poincaré in the 1880s as the monodromy groups of certain 2^{nd} order differential equations on the complex plane $\mathbb{C}$. These are, classically, discrete subgroups of $\mathrm{PSL}(2, \mathbb{C})$, the group of holomorphic automorphisms of the complex projective line $\mathbb{P}^1_{\mathbb{C}}$, which act on this space with nonempty region of discontinuity. Equivalently, these can be regarded as groups of conformal automorphisms of the sphere $\mathbb{S}^2$, or as groups of (orientation preserving) isometries of the hyperbolic 3-space.

Kleinian groups have played for decades a major role in several fields of mathematics, as for example in Riemann surfaces and Teichmüller theory, automorphic forms, holomorphic dynamics, conformal and hyperbolic geometry, 3-manifolds theory, iteration theory of rational maps, etc.

Much of the theory of Kleinian groups has been generalised to conformal Kleinian groups in higher dimensions (also called *Möbius* or *hyperbolic* Kleinian groups), i.e., discrete groups of conformal automorphisms of the sphere $\mathbb{S}^n$. We refer to [105, 106] for clear accounts on conformal Kleinian groups.

Also, D. P. Sullivan's dictionary gives remarkable relations between classical Kleinian groups and the iteration theory of rational maps on $\mathbb{P}^1_{\mathbb{C}}$. Many interesting results about the dynamics of rational maps on $\mathbb{P}^1_{\mathbb{C}}$ in the last decades have been motivated by the dynamics of Kleinian groups (see for instance the references to Sullivan's and McMullen's work in the bibliography). Further striking relations in this sense have been also obtained by M. Lyubich, Y. Minsky and others.

On the other hand, the iteration theory of rational maps is being generalised to higher dimensions by various authors, like J. F. Fornæss, E. Bedford, J. Smillie, N. Sibony, amongst others, obtaining many interesting results about the dynamics of rational endomorphisms of $\mathbb{P}^n_{\mathbb{C}}$ (see [51] for a clear account on the subject).

It is thus natural to ask: what about the other side of the dictionary, in holomorphic dynamics, in higher dimensions? I.e., what about discrete groups of automorphisms of $\mathbb{P}^n_{\mathbb{C}}$? That is the subject we explore in this monograph.

As we will see in the sequel, this includes the theory of discrete groups of isometries in both, real and complex hyperbolic geometry, as well as discrete complex affine groups. There are several other ways and sources from which complex Kleinian groups arise, and we discuss some of these throughout the text.

This monograph originates in articles about *complex Kleinian groups* by A. Verjovsky, J. Seade, J. P. Navarrete and A. Cano (see references in the bibliography), enriched through the work of many authors who have studied and written about real and complex hyperbolic geometry, and also about holomorphic projective structures on complex manifolds. We have particularly profited from W. Goldman's excellent book on complex hyperbolic geometry, as well as John Parker's articles cited in our bibliography. Misha Kapovich's work has also been very helpful and highly inspiring. These and many other works have played a significant role for us while writing this monograph; throughout the text we indicate the most significant references on each topic, and each chapter begins with an introduction that includes references for further reading.

A complex Kleinian group means a discrete group of automorphisms of $\mathbb{P}^n_{\mathbb{C}}$ which acts within a nonempty region of discontinuity. When the group acts on $\mathbb{P}^n_{\mathbb{C}}$ preserving a ball, then it is conjugate to a subgroup of $\mathrm{PU}(n,1)$ and we are in the framework of complex hyperbolic geometry; the groups one gets in this way are called *complex hyperbolic Kleinian groups*. If the group acts on $\mathbb{P}^n_{\mathbb{C}}$ preserving a projective hyperplane $\mathbb{P}^{n-1}_{\mathbb{C}}$, then we are essentially in the realm of complex affine geometry. On the other hand, whenever a discrete subgroup Γ of $\mathrm{PSL}(n+1,\mathbb{C})$ acts properly discontinuously on an invariant open set $\Omega \subset \mathbb{P}^n_{\mathbb{C}}$, the quotient space Ω/Γ is an orbifold equipped with a projective structure, and the study of holomorphic projective structures on complex manifolds and orbifolds is in itself a rich area of current research. This includes the theory of complex hyperbolic and complex affine manifolds. Thence the theory of complex Kleinian groups provides a means to study these important fields of mathematics in a unified way.

The study of complex Kleinian groups is indeed still in its childhood, and this monograph aims to contribute to the laying down of its foundations, studying basic concepts as for instance that of the limit set; constructions of discrete groups in higher dimensions; the uniformisation problem for two-dimensional complex orbifolds; classification problems: of the elements in $\mathrm{PSL}(n+1,\mathbb{C})$ and their geometry and dynamics, of its discrete subgroups, of the complex structures one gets on quotients of open sets of $\mathbb{P}^n_{\mathbb{C}}$ which are invariant under the action of a discrete group; Teichmüller theory in higher dimensions; relations with twistor theory; etc.

Whenever one has a classical Kleinian group, one has a natural splitting of $\mathbb{P}^1_{\mathbb{C}}$ in two invariant subsets: one of these, say Ω, is where the action is discontinuous; this is also the equicontinuity set of the group, i.e., the points where the group forms a normal family. The other set Λ, its complement, is where the dynamics "concentrates". The set Ω plays a key role in complex geometry, as shown by the work of Ahlfors, Kra, Bers and many others. And the action on the limit set plays a key role for holomorphic dynamics, as shown by the work of D. Sullivan, W. Thurston, C. McMullen, M. Lyubich and many others.

One has a similar picture for conformal Kleinian groups in higher dimensions, and their study is the content of Chapter 1 of this monograph, which contains well-known material that we present in a way that supports the remaining chapters.

We refer to the literature, particularly to the recent articles of M. Kapovich listed in the bibliography, for a deeper and wider study of this important field of mathematics.

Chapter 1 provides a quick glance at the foundations of real hyperbolic geometry: its various models; its group of isometries; the relation between hyperbolic space and conformal geometry on the sphere at infinity; discrete groups of isometries; the limit set and discontinuity region; fundamental domains. The literature on these topics is vast, so we have made no attempt to give a comprehensive account of the subject. Instead, we give a number of examples and discuss some of the key ideas that help to get a feeling for this exciting subject. We also explain briefly some of the most celebrated theorems in the subject that we use in the sequel, such as Moore's ergodicity theorem, Mostow's rigidity, the Patterson-Sullivan measure, the ergodicity of action on the limit set, and Sullivan's theorem of nonexistence on invariant line-fields on the limit set.

In Chapter 2 we look at complex hyperbolic geometry, which is the complex analogue of real hyperbolic geometry. Its origin traces back to the work of É. Picard on differential equations in several complex variables. Later, G. Giraud made fundamental contributions to the subject through a series of papers, which are discussed in an appendix at the end of [67]. This is a fascinating branch of mathematics which has been having a fast development over the last few decades, thanks to the contributions of many authors as, notably, G. Mostow and P. Deligne, and more recently W. Goldman, J. Parker, N. Gusevskii, R. Schwartz, E. Falbel, M. Kapovich and several others. In this chapter we briefly describe the classical models for complex hyperbolic geometry, its group of holomorphic isometries, which is the projective Lorentz group $\mathrm{PU}(n,1)$, and methods for constructing discrete subgroups of it. We describe, following [67], the classification of the elements in $\mathrm{PU}(n,1)$ according to their geometry and dynamics, and for $n=2$ also according to trace. We finish the chapter by giving the definition and basic properties of the limit set following [45].

In Chapter 3 we start our discussion on complex Kleinian groups. As mentioned earlier, these are, by definition, discrete subgroups of $\mathrm{PSL}(n+1,\mathbb{C})$ that act on $\mathbb{P}^n_{\mathbb{C}}$ with nonempty regions of discontinuity. This naturally includes the discrete groups of isometries of real and complex hyperbolic n-space, as well as all complex affine groups, and many more. In this general setting there is no well-defined notion of limit set when $n \geq 2$. We give an example in $\mathbb{P}^2_{\mathbb{C}}$, motivated by work of R. Kulkarni in the late 1970s, that illustrates the diversity of possibilities one has for defining this notion, unlike the situation in hyperbolic geometry where "the limit set" is a well-defined notion. There are several possible definitions of this concept, each with its own properties and characteristics. There is the Chen-Greenberg limit set for complex hyperbolic groups, i.e., subgroups of $\mathrm{PU}(n,1)$; there is the Kulkarni limit set; there is the complement of the region of discontinuity; the complements of the maximal regions where the action is properly discontinuous (in general there is no largest such region); and the complement of the region of equicontinuity. There are also other sets that can play the role of "the limit set" in different settings (e.g. the closure of the fixed points of loxodromic elements, whenever this makes sense). These notions are not always equivalent, as we explain in the text, and each of them has its own interest. Yet, in all cases one has a region Ω where the dynamics is "tame", and a set Λ where the dynamics concentrates.

When $n = 1$ a complex Kleinian group is nothing but a discrete group of automorphisms of $\mathbb{P}^1_{\mathbb{C}}$, which is the Riemann sphere. A natural next step is considering groups of automorphisms of $\mathbb{P}^2_{\mathbb{C}}$ and this is our focus of study in Chapters 4 to 8. The material in these chapters is mostly based on work done by Angel Cano and Juan-Pablo Navarrete, and more recently also in collaboration with Waldemar Barrera.

In Chapter 4 we look at the geometry and dynamics of the individual elements in $\mathrm{PSL}(3,\mathbb{C})$. In other words, we consider a single automorphism of $\mathbb{P}^2_{\mathbb{C}}$ and the cyclic group it generates. We know that, classically, the elements in $\mathrm{PSL}(2,\mathbb{C})$ are of three types: elliptic, parabolic and loxodromic (or hyperbolic). This classification is done in terms of their geometry and dynamics, and it can also be done algebraically, in terms of the trace of a lifting to $\mathrm{SL}(2,\mathbb{C})$. In fact there is a similar (geometric-dynamical) classification for the isometries of every Riemannian manifold of nonpositive curvature, and even more generally for isometries of CAT(0)-spaces (see Remark 2.4.6). In Chapter 4 we show that the elements of $\mathrm{PSL}(3,\mathbb{C})$ can be also naturally classified into these three types, elliptic, parabolic and loxodromic, although $\mathbb{P}^2_{\mathbb{C}}$ has positive curvature and the action of $\mathrm{PSL}(3,\mathbb{C})$ is not by isometries. This classification is of course compatible with the classical one when the elements lie in $\mathrm{PU}(2,1)$. The classification is done in terms of the geometry and dynamics, and also algebraically. It turns out that all the elliptic and parabolic elements are actually conjugate to elliptic and parabolic elements in $\mathrm{PU}(2,1)$. The new phenomena appear when looking at the loxodromic elements. We remark that the geometric–dynamical classification of the elements in $\mathrm{PSL}(3,\mathbb{C})$ actually extends to higher dimensions, as shown in [39].

The next step in our study is considering in Chapter 5 subgroups of $\mathrm{PSL}(3,\mathbb{C})$ whose dynamics is governed by a subgroup of $\mathrm{PSL}(2,\mathbb{C})$. This is natural since the subgroups of $\mathrm{PSL}(2,\mathbb{C})$ are far better understood than the subgroups of $\mathrm{PSL}(3,\mathbb{C})$. The simplest way for doing so is by taking a discrete group Γ in $\mathrm{PSL}(2,\mathbb{C})$, lifting it to $\mathrm{SL}(2,\mathbb{C})$ and looking at its canonical inclusion in $\mathrm{SL}(3,\mathbb{C})$. This type of groups were called *suspensions* in [201] and this construction was generalised in [160], [41], getting interesting representations in $\mathrm{PSL}(3,\mathbb{C})$ of subgroups of $\mathrm{PSL}(2,\mathbb{C})$. Yet, there are many other situations in which one can get a lot of information about a given group in $\mathrm{PSL}(3,\mathbb{C})$ from a lower-dimensional one which "controls" its dynamics. This is the topic we study in Chapter 5. It is worth remarking that even if we start with a discrete group in $\mathrm{PSL}(3,\mathbb{C})$, it can happen that the corresponding control group in $\mathrm{PSL}(2,\mathbb{C})$ is nondiscrete. We thus make in this chapter a brief discussion of nondiscrete subgroups of $\mathrm{PSL}(2,\mathbb{C})$. Of course it makes no sense in this setting to speak of a "discontinuity region", since this is empty by definition. Here, it is the equicontinuity region which plays a significant role.

As we have said before, unlike the situation for subgroups of $\mathrm{PSL}(2,\mathbb{R})$, in higher dimensions there is no unique notion of "the limit set" for complex Kleinian groups. There are instead several natural such notions, each with its own properties and characteristics, each providing a different kind of information about the geometry and dynamics of the group. Yet, in Chapter 6 we see that in dimension 2, the various natural definitions of limit set coincide generically, i.e., for complex Kleinian groups whose Kulkarni limit set has "enough" lines.

This is interesting also from the viewpoint of having a Sullivan dictionary between Kleinian groups and iteration theory in several complex variables. In fact we recall the important theorem in [60]), stating that in the space of all rational maps of degree d in $\mathbb{P}^n_{\mathbb{C}}$, for $n \geq 2$, those whose Fatou set is Kobayashi hyperbolic form an open dense set with the Zariski topology. Similarly, in this chapter we see that under certain "generic" conditions, the region of equicontinuity of a complex Kleinian group in $\mathbb{P}^2_{\mathbb{C}}$ coincides with the Kulkarni region of discontinuity, and it is the largest open invariant set where the group acts properly discontinuously. And this region is Kobayashi hyperbolic. Hence the results in this chapter, which are based on work by W. Barrera, A. Cano and J. P. Navarrete, give us a better understanding of the concept of "the limit set" in dimension 2, and they set down the first steps of a theory that points towards an analogous concept for Kleinian groups of the aforementioned theorem of Fornæss-Sibony.

In Chapter 7 we consider again complex hyperbolic Kleinian groups, i.e., discrete subgroups of $\mathrm{PU}(n,1)$, but we now look at their action on the whole projective space $\mathbb{P}^n_{\mathbb{C}}$, not only at the projective ball that serves as a model for complex hyperbolic space. We know from Chapter 3 that one has in this setting several possible definitions of the limit set. Here we compare the Kulkarni limit set and the complement of the region of equicontinuity, with the limit set in the sense of Chen-Greenberg. This is a subset of the sphere that bounds the projective ball that serves as a model for the complex hyperbolic space $\mathbb{H}^n_{\mathbb{C}}$. This allows us to get information about the action of the group on all of $\mathbb{P}^n_{\mathbb{C}}$ from its behaviour on the ball $\mathbb{H}^n_{\mathbb{C}}$.

In Chapter 8 we bring together the information obtained through Chapters 4, 5 and 7, to study discrete subgroups of $\mathrm{PSL}(3,\mathbb{C})$ with a divisible set in $\mathbb{P}^2_{\mathbb{C}}$ in the sense of Y. Benoist. More precisely, we study subgroups of $\mathrm{PSL}(3,\mathbb{C})$ acting on $\mathbb{P}^2_{\mathbb{C}}$ so that there is a nonempty open invariant set Ω where the group acts properly discontinuously and the quotient $M = \Omega/\Gamma$ is compact; we call such actions *quasi-cocompact*. This includes the cocompact case: When Γ has a largest region of discontinuity and the quotient is compact. The surface M is an orbifold naturally equipped with a projective structure. The material in this chapter is closely related to previous work by S. Kobayashi, T. Ochiai, Y. Inoue, B. Klingler and others, about compact complex surfaces with a projective structure. We give the complete classification of the divisible sets appearing in this way, the corresponding groups, the Kulkarni limit set and the topology of the quotient orbifold.

Chapters 9 and 10 focus on complex Kleinian groups in higher dimensions, and these are essentially based on the articles by José Seade and Alberto Verjovsky listed in the bibliography. The material in Chapter 9 is actually related to previous work by M. Nori aimed at construction of new compact complex manifolds. We also speak in that chapter about work by A. Cano.

Recall that the classical Schottky groups are subgroups of $\mathrm{PSL}(2,\mathbb{C})$ obtained by considering disjoint families of circles in $\mathbb{P}^1_{\mathbb{C}} \cong \mathbb{S}^2$. These circles play the role of *mirrors* that split the sphere in two diffeomorphic halves which are interchanged by a conformal map, and these maps generate the Schottky group (see Chapter 1 for details). So the idea of constructing Schottky groups in higher dimensions is to construct *mirrors* in $\mathbb{P}^n_{\mathbb{C}}$ that split the space in two parts which are interchanged by a holomorphic authomorphism, and use these to construct discrete subgroups. This works fine on odd-dimensional projective spaces. One gets Schottky subgroups of $\mathrm{PSL}(2n+2,\mathbb{C})$ whose limit sets are solenoids with rich dynamics. Following [203], we determine the topology of the compact complex manifolds obtained as quotient $M_{\check{\Gamma}} := \Omega(\check{\Gamma})/\check{\Gamma}$ of the region of discontinuity divided by the action. We look at their Kuranishi space of versal deformations and prove that, for $n > 2$, every infinitesimal deformation of $M_{\check{\Gamma}}$ actually corresponds to an infinitesimal deformation of the group $\check{\Gamma}$ in the projective group $\mathrm{PSL}(2n+2,\mathbb{C})$. This is analogous to the classical Teichmüller theory for Riemann surfaces. Similar considerations were observed in [176] for $n = 1$, studying the so-called Pretzel Twistor spaces.

In even dimensions, one can show that there cannot be Schottky groups (see the text for a more precise statement). Yet, in these dimensions one does have mirrors, but unlike the odd-dimensional case, mirrors are now singular varieties, not smooth submanifolds, and all mirrors must intersect. Thus one gets *kissing-Schottky* groups as in the "Indra's Pearls" of [158], acting on $\mathbb{P}^2_{\mathbb{C}}$ and actually on all projective spaces. The examples of kissing-Schottky groups that we give in Chapter 9 are interesting because these groups are not elementary (see the text for the definition), nor affine, nor complex hyperbolic, in contrast with the groups that appear in Chapter 8.

Finally, Chapter 10 is based on [202]. Here we use twistor theory to construct complex Kleinian groups. Twistor theory is, no doubt, one of the jewels of mathematics in the 20th Century.

There are two different ways in which twistor theory can be considered. One is to see twistor theory as providing the geometrical setting for new and valuable mathematical methods in, for example, the treatment of Yang-Mills and other nonlinear equations. The other point of view, more ambitious, is the twistor programme for physics, in which it is held that, if the nature of the physical world is to be understood, then the usual description of space-time must be superseded by some form of twistor geometry. The mathematical foundations of this theory were developed by R. Penrose and a number of other great mathematicians such as M. Atiyah, N. Hitchin and others.

The "Penrose twistor programme" springs from the remarkable fact that there is a rich interplay between the conformal geometry of (even-dimensional) Riemannian manifolds and the complex geometry of their twistor spaces. In Chapter 10 we explain how this interplay can be also "pushed forward" to dynamics, providing interesting relations between conformal and holomorphic dynamics. One gets that every conformal Kleinian group can be regarded, canonically, as a complex Kleinian group via twistor theory. For example in dimension 4, one has a canonical embedding $\mathrm{Conf}_+(\mathbb{S}^4) \hookrightarrow \mathrm{PSL}(4, \mathbb{C})$ and the dynamics of conformal Kleinian groups in dimension 4 embeds in the dynamics of complex Kleinian groups in $\mathbb{P}^3_{\mathbb{C}}$. Furthermore, one can actually see things about conformal Kleinian groups $G \subset \mathrm{Conf}_+(\mathbb{S}^4)$ through their action on $\mathbb{P}^3_{\mathbb{C}}$ which are not visible through their action on $\mathbb{S}^4$. Similar statements hold in higher dimensions.

As mentioned before, the theory of complex Kleinian groups is still in its childhood, although its origin traces back to the work of Riemann, Poincaré, Klein, Picard, Giraud and many other great mathematicians. The knowledge we now have about classical Kleinian groups, as well as about complex hyperbolic groups, complex affine groups, geometric structures on complex manifolds, moduli spaces and Teichmüller theory, potential theory and iteration theory in several complex variables, inspire many questions and lines of further research in the topic of discrete subgroups of projective transformations, that are waiting to be explored.

Although the various chapters in this monograph add up to form a coherent unit, this monograph has been written so that each individual chapter can be read on its own. No doubt the theory of complex Kleinian groups will eventually become an important subfield of both, complex geometry and holomorphic dynamics, and we hope this monograph will help to lay down its foundations, and contribute to enhancing the interest in this fascinating topic.

Acknowledgments

This monograph is the outcome of several years of working in this beautiful area of mathematics. During this time we have profited from conversations with many friends and great mathematicians, to whom we are deeply indebted: Our heart-felt thanks to all of them. Although we are sure to leave out some important names, we want to mention here those we are able to remember.

First of all, we are much indebted to Alberto Verjovsky, for explaining to us many of the ideas and concepts included in this text, and for teaching us about the beauty of mathematics.

We are most grateful to professors Dennis Sullivan, John Erick Fornæss, Nessim Sibony, William Goldman, Nikolay Gusevskii, Misha Kapovich, Etienne Ghys, Misha Lyubich, Paulo Sad and John Parker. We have learned a lot from their works, their lectures, their conversations and their way of thinking. These have been to us like lights along the path. Our deepest thanks also to our colleagues Waldemar Barrera, Carlos Cabrera, Aubin Arroyo, Adolfo Guillot, Luis Loeza and Piotr Makienko. Each of them has contributed in one way or another to this monograph.

During this time we have also profited from the support of several institutions which have either hosted us or given us financial support, or both. Our deepest thanks to all of them, though we can only mention here a few of these. First of all, we are indebted to the Instituto de Matemáticas of the National University of Mexico (UNAM) and its Cuernavaca Unit, where two of us work, as well as to the Facultad de Matemáticas of the Universidad Autónoma de Yucatán (UADY), the home institution of the third one of us. Both of these institutions, and the people in them, have supported us in all ways.

We are most grateful to IMPA at Rio de Janeiro, Brazil, which always is like a home for us, offering us a most stimulating academic atmosphere; and to the Abdus Salam International Centre for Theoretical Physics at Trieste, Italy, where we have spent long periods of time, and where we profited from the School and Workshop “Discrete groups in Complex Geometry” in 2010, that left a mark on all of us.

During these years we have had several grants and support from the DGAPA of the National University of Mexico (UNAM), as well as from CONACYT, México, the CNRS, France, mostly through the Laboratoire International Associé Solomon Lefschetz (LAISLA).