Birkhäuser

Monographs in Mathematics
Vol. 103

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Methods of Geometric Analysis in Extension and Trace Problems

Volume 2

Birkhäuser

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2010 Mathematics Subject Classification: 26A16, 26B35, 46B85, 46B70, 51H25, 52A07, 53C23, 54E35, 54E40

ISBN 978-3-0348-0211-6 e-ISBN 978-3-0348-0212-3
DOI 10.1007/978-3-0348-0212-3

Library of Congress Control Number: 2011939775

Printed on acid-free paper

Springer Basel AG is part of Springer Science+Business Media

www.birkhauser-science.com

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Preface

The volume contains an exposition of the recent development of the two main themes of the book, Lipschitz extension problems for maps between metric spaces and smooth extension-trace problems for functions on closed subsets of $\mathbb{R}^n$. The basic facts used for their study are presented in Volume I while the current volume consists only of detailed motivations and formulations within the corresponding proofs. The reader may find in the introduction of each chapter below a detailed description of the material of that chapter. Here we restrict ourselves to a discussion of the main features of the two parts forming the volume.

The first of the aforementioned topics is presented in Part 3 consisting of Chapters 6-8. Chapter 6 is mainly devoted to the Lang-Schlichenmaier theory relating the existence of the corresponding Lipschitz extensions to two important concepts of Lipschitz topology, *Nagata dimension* and *Lipschitz connectedness*, see Section 4.2 of Volume I. This culminates with an explicit construction of Lipschitz extension operators acting between large classes of metric spaces. If the target space in question is Banach, the corresponding operator becomes linear, i.e., it provides the simultaneous Lipschitz extension for all subsets of the domain. However, the extension constants in the latter case are either unspecified or have only coarse estimates of how they depend on the basic parameters.

In Chapter 7, we present two other methods for the Lipschitz simultaneous extension problem which give the corresponding results with extension constants close to optimal. One of them, due to Lee and Naor, exploits a probabilistic argument that first appeared in a computer science context, see, e.g., Section 4.1 of Volume I. A second, due to the authors of this book, is based on geometric analysis methods; this method is constructive and covers an essentially wider class of spaces.

Finally, Chapter 8 studies different relations between linear and nonlinear Lipschitz extensions and the corresponding extension constants. In particular, we study the influence of snowflake metric transforms on the existence of Lipschitz extensions, present an explicit formula relating the linear Lipschitz extension constants with the corresponding nonlinear ones and construct examples of metric spaces with Lipschitz extension constants that are finite for the nonlinear case but infinite for the linear case.

Finally, Part 4 of Volume II consisting of Chapters 9 and 10 is devoted to the smooth extension problems for functions on closed subsets of $\mathbb{R}^n$.

The first section of Chapter 9 presents the Yu. Brudnyi–Shvartsman characterization of the trace spaces to closed subsets for Lipschitz spaces of higher order and for the associated jet spaces. The proofs are strongly based on concepts and methods of Local Approximation Theory, see Section 2.3 and Appendices F, G of Volume I for some basic facts of the theory. This approach used throughout the chapter gives solutions to similar problems for Morrey-Campanato and higher order Lipschitz spaces and for a wide class of closed subsets of $\mathbb{R}^n$. This class, in particular, includes Ahlfors s-regular sets with $s > n-1$, some self-similar fractals with separation conditions, see Section 4.2 of Volume I, and the closure of *uniform* (or $\varepsilon - \delta$) domains, see Section 2.4 of Volume I.

Chapter 10 discusses two extension-trace problems going back to the classical 1934 Whitney papers. The first one asks how to distinguish traces to a closed set of functions from the space $C^k(\mathbb{R}^n)$ or the likes from those of other continuous functions on this set (*Whitney's trace problem*). The solution may be essentially simplified if one reduces the problem to the case of subsets containing only a fixed number of points depending only on parameters of the space in question (e.g., subsets of at most $3 \cdot 2^{n-1}$ points for $C^{1,1}(\mathbb{R}^n)$). This reduction is the direct consequence of the general Yu. Brudnyi-Shvartsman *finiteness principle* proved up to now only in some special cases, see subsections 10.3.2, 10.4.1 and 10.5.1 of this chapter.

The second Whitney problem asks about the existence of a linear continuous extension operator for the trace of a smoothness space into the space itself. The problem is solved constructively in several cases, see subsections 10.2.4, 10.2.5, 10.4.2 and 10.5.2 of this chapter.

All the above mentioned results are carried out in accordance with the geometric analysis approach of this book combining Lipschitz selection theorems of Sections 5.4 and 5.5 of Volume I with the local approximation extension-trace criteria of Sections 9.3 and 9.4 of this volume.

A completely new approach to the Whitney problems for the spaces $C^\ell(\mathbb{R}^n)$ and $C^{\ell,\omega}(\mathbb{R}^n)$ was proposed by Ch. Fefferman in a series of papers starting in 2003 and continuing to the present. We present a detailed account of his breakthrough results with a rather sketchy description of his methods in Sections 10.3 and 10.4. A deeper insight into Fefferman's methods is, from our point of view, the most actual problem in this area. Clarifying these very complicated and lengthy proofs and the ideas behind them surely will lead to important progress in the area.

Basic Terms and Notation

Set-theoretic operations

$\in$ membership

$\cup$ union

$\cap$ intersection

$\setminus$ set theoretic difference

$\subset$ embedding (not necessarily proper)

$\oplus$ direct sum (also known as direct or cartesian product)

Sets and subsets

Let $T \subset S$ be sets.

- $T := \{x \in S\,;\, \mathcal{P}\}$: all elements of the subset T have the property $\mathcal{P}$

  Note the *figured brackets* and *semicolon* designed for the notation of sets. Hereafter the symbol $:=$ means that the left-hand side is defined or denoted by the right-hand side.

- $T^c := \{x \in S\,;\, x \notin T\}$: complement of T in S; it is usually clear from the context with respect to which larger set S the complement is taken

- 2^S: collection of subsets in S

- Let $\mathcal{F} := \{S_j\}_{j\in J} \subset 2^S$ be a *family* (indexed set). Then

$$\cup\,\mathcal{F} := \bigcup_{j\in J} S_j, \qquad \cap\,\mathcal{F} := \bigcap_{j\in J} S_j,$$

  $\sqcup\,\mathcal{F}$: disjoint union (($S_j \cap S_{j'} = \emptyset$ if $j \neq j'$ in this case)

- $\mathcal{F}$ is a *cover* of S if $\cup\mathcal{F} = S$

- a cover $\mathcal{F}'$ is a *refinement* of $\mathcal{F}$ if every $S' \in \mathcal{F}'$ is a subset of some $S \in \mathcal{F}$

Functions

Let S, S' be sets and $T \subset S$, $T' \subset S'$.

- $f : S \to S'$: function (map, transform) acting from S into S'
- $x \mapsto f(x)$: the alternative notation of f whenever S, S' are clear from the context
- $\mathrm{Im}\, f := \{f(x) \in S'\,;\, x \in S\}$: image (range) of f
- $f(S)$: the alternative notation of $\mathrm{Im}\, f$
- $f^{-1}(T') := \{x \in S\,;\, f(x) \in T'\}$: coimage of $T' \subset S'$
- $f^{-1} : \mathrm{Im}\, f \to 2^S$: inverse to f given by $x' \mapsto f^{-1}(\{x'\})$
- $f|_T : T \to S'$: trace (restriction) of f to $T \subset S$

Let $F : S \to 2^{S'} \setminus \{\emptyset\}$ be a set-valued (multivalued) map.

- $f : S \to S'$: *selection* of F if $f(x) \in F(x)$ for all x
- $\mathrm{card} : 2^S \to [0, +\infty]$: cardinality (number of points)
- $\mathrm{ord}\, \mathcal{F}$: *order (multiplicity)* of $\mathcal{F}$, i.e., $\mathrm{ord}\, \mathcal{F} := \sup_{x \in S} \big(\mathrm{card}\, \{j \in J\}\,;\, S_j \ni x\}\big)$
- $\mathbf{1}_T : S \to \{0, 1\}$: indicator (characteristic function) of $T \subset S$

Numbers and related vector spaces

- $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$: natural, integer, rational and real numbers
- $\mathbb{Z}_+, \mathbb{R}_+$: nonnegative integers and real numbers
- (a, b), $[a, b]$: open and closed intervals with endpoints $a, b \in \mathbb{R} \cup \{-\infty, \infty\}$
- $\{e_j\}_{1 \le j \le n}$: the standard basis of $\mathbb{R}^n$
- $\langle x, y \rangle := \sum_{i=1}^{n} x_i y_i$: the standard scalar product (sometimes also denoted by $x \cdot y$)
- $x \mapsto \|x\|_p := \left\{ \sum_{i=1}^{n} |x_i|^p \right\}^{1/p}$: ℓ_p-norm (quasinorm if $0 < p < 1$)
- $\ell_p^n := (\mathbb{R}^n\,, \|\cdot\|_p)$

Subsets of $\mathbb{R}^n$

- $\mathbb{Z}^n := \{x \in \mathbb{R}^n\,;\, x_i \in \mathbb{Z} \text{ for all } i\}$
- $\mathbb{Z}_+^n := \{x \in \mathbb{Z}^n\,;\, x_i \ge 0 \text{ for all } i\}$

- α, β, γ: elements of $\mathbb{Z}_+^n$
- $\mathbb{S}^{n-1} := \{x \in \mathbb{R}^n\,;\, \|x\|_2 = 1\}$: the unit sphere
- $(x, y), [x, y]$: open and closed intervals with endpoints $x, y \in \mathbb{R}^n$
- $Lin(\mathbb{R}^n)$, $\mathrm{Aff}(\mathbb{R}^n)$: the sets of linear and affine subspaces in $\mathbb{R}^n$
- $\mathcal{C}(\mathbb{R}^n)$: the set of nonempty bounded convex sets in $\mathbb{R}^n$
- hull: *linear hull* (span, envelope)
- aff: *affine hull*
- conv: *convex hull*
- $Q_r(x)$ (briefly Q, Q' etc.): closed cube (ℓ_∞^n ball) in $\mathbb{R}^n$ of center x and radius $r > 0$
- c_Q, r_Q: the center and radius of a cube Q
- $\mathcal{K}(\mathbb{R}^n)$: the set of closed cubes in $\mathbb{R}^n$
- $\mathcal{K}_S := \{Q_r(x) \in \mathcal{K}(\mathbb{R}^n)\,;\, x \in S \text{ and } 0 < r \le 2\,\mathrm{diam}\, S\}$
- $\mathcal{W}_S$: Whitney cover of S^c for a closed subset $S \subset \mathbb{R}^n$

Polynomials, derivatives, differences

Let $\alpha, \beta \in \mathbb{Z}_+^n$.

- $x \mapsto x^\alpha := \prod_{i=1}^n x_i^{\alpha_i}$, $x \in \mathbb{R}^n$: α-monomial (stipulation: $0^0 := 1$)
- $|\alpha| := \sum_{i=1}^n \alpha_i, \quad \alpha! := \prod_{i=1}^n \alpha_i!, \quad \binom{\alpha}{\beta} := \frac{\alpha!}{(\alpha-\beta)!\beta!}$
- $\mathcal{P}_{k,n}$: the space of polynomials in $x \in \mathbb{R}^n$ of degree k, the linear hull of α-monomials with $|\alpha| \le k$
- $D_i := \frac{\partial}{\partial x_i}$, $1 \le i \le n$: the i-th partial derivative
- $D_x := \sum_{i=1}^n x_i D_i$: derivative in direction $x \in \mathbb{S}^{n-1}$
- $\nabla := (D_1, \ldots, D_n)$: gradient
- $D^\alpha := \prod_{i=1}^n D_i^{\alpha_i}$: mixed α-derivative

Let f be k-times differentiable at $x \in \mathbb{R}^n$.

- $T_x^k f := \sum_{|\alpha| \le k} \frac{(\cdot - x)^\alpha}{\alpha!} D^\alpha f(x)$: Taylor's polynomial at x of degree k

Let $f : \mathbb{R}^n \to \mathbb{R}$ and $h \in \mathbb{R}^n$.

- $\tau_h f := f(\cdot + h)$: h-shift
- $\Delta_h^k := (\tau_h - 1)^k = \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} \tau_{jh}$: k-difference of step h
- $\Delta_h^\alpha := \prod_{i=1}^{n} \Delta_{h_i e_i}^{\alpha_i}$: (mixed) α-difference

Topological and metric spaces

Let S be a subset of a Hausdorff topological space.

- $\bar{S}$: closure, S^o: interior, $\partial S := \bar{S} \cap \bar{S}^c$: boundary
- $(\mathcal{M}, d)$: metric space with underlying set $\mathcal{M}$ and metric d (briefly, $\mathcal{M}$ whenever d is clear from the context)

  Throughout the book $\mathcal{M}$ is assumed to be nontrivial, i.e., $\operatorname{card} \mathcal{M} > 1$.
- m, m' etc.: points of $\mathcal{M}$
- $S \subset (\mathcal{M}, d)$ (briefly, $S \subset \mathcal{M}$): a metric subspace of $(\mathcal{M}, d)$
- $(\mathcal{M}, m_0, d)$: punctured metric space ($m_0 \in \mathcal{M}$)
- $S \subset (\mathcal{M}, m_0, d)$: a metric subspace of the punctured metric space (i.e., $m_0 \in S$)
- $B_r(m_0)$, $\bar{B}_r(m_0)$: open and closed balls in $\mathcal{M}$ of *center* x_0 and radius $r > 0$;

  $\bar{B}_r(m_0) := \{m \in \mathcal{M}\,;\, d(m, m_0) \le r\}$ does not, in general, coincide with the closure $\overline{B_r(m_0)}$ of $B_r(m_0) := \{m \in \mathcal{M}\,;\, d(m, m_0) < r\}$

Let S, S' be subsets of $(\mathcal{M}, d)$.

- $S_\varepsilon := \cup\{B_\varepsilon(m) \subset \mathcal{M}\,;\, m \in S\}$: ε-neghborhood of S
- $d(m, S) := \inf\{d(m, m')\,;\, m' \in S\}$: distance from m to S
- $Pr_S : \mathcal{M} \to 2^S$: metric projection onto S, i.e.,

$$Pr_S(m) := \{m' \in S\,;\, d(m, m') = d(m, S)\}$$

- $d(S, S') := \inf\{d(m, m')\,;\, (m, m') \in S \oplus S'\}$: distance between S and S'

- $d_{\mathcal{H}}(S, S') := \inf\{\varepsilon > 0\,;\ S \subset S'_\varepsilon,\ S' \subset S_\varepsilon\}$: Hausdorff distance between S and S'

Lipschitz functions

Let $f : (\mathcal{M}, d) \to (\mathcal{M}', d')$.

- $L(f; \mathcal{M}, \mathcal{M}') := \sup\limits_{m_1 \neq m_2} \dfrac{d'(f(m_1), f)m_2))}{d(m_1, m_2)}$: Lipschitz constant (briefly, $L(f)$)
- $|f|_{Lip(\mathcal{M},\mathcal{M}')}$: alternative notation for $L(f)$
- f is C*-Lipschitz*, if $L(f) \le C$ and f is *Lipschitz* if $L(f)$ is finite
- f is C*-bi-Lipschitz embedding*, if f^{-1} exists and its *distortion* $D(f)$ satisfies $D(f) := \max\{L(f), L(f^{-1})\} \le C$
- f is a C*-isometry* (isometry for $C = 1$) if f is a bijection with $D(f) \le C$
- f is a *bi-Lipschitz homeomorphism* if f is a C-isometry for some C

Continuous and Lipschitz spaces

Let $\mathcal{M}, \mathcal{M}'$ be metric spaces.

- $C(\mathcal{M})$: the space of real continuous functions
- $C_u(\mathcal{M})$: the space of real uniformly continuous functions
- $C_b(\mathcal{M})$: the space of real bounded continuous functions equipped with the uniform norm
- $Lip(\mathcal{M}, \mathcal{M}')$: the space of Lipschitz maps from $\mathcal{M}$ into $\mathcal{M}'$ equipped with the seminorm $f \mapsto L(f)$
- $Lip(\mathcal{M}) := Lip(\mathcal{M}, \mathbb{R})$
- $Lip(\mathcal{M}, m_0, \mathbb{R}^n) := \{f \in Lip(\mathcal{M}, \mathbb{R}^n)\,;\ f(m_0) = 0\}$ (briefly, $Lip_0(\mathcal{M}, \mathbb{R}^n)$ if the choice of m_0 is clear)

Let $G \subset \mathbb{R}^n$ be a *domain* (open connected set) and ω belongs to the class of k*-majorant* Ω_k:

- $t \mapsto \omega_k(t; f)_G$, $t > 0$: k-modulus of continuity of $f : G \to \mathbb{R}$ (the subindex G is omitted for $G = \mathbb{R}^n$)
- $\dot{\Lambda}^{k,\omega}(G)$: the "homogeneous" space of k-Lipschitz functions on G equipped with the seminorm $f \mapsto |f|_{\Lambda^{k,\omega}(G)} := \sup\limits_{t>0} \dfrac{\omega_k(t; f)_G}{\omega(t)}$
- $\Lambda^{k,\omega}(G) \subset \dot{\Lambda}^{k,\omega}(G)$: the "nonhomogeneous" space of k-Lipschitz functions on G equipped with the norm $f \mapsto \|f\|_{\Lambda^{k,\omega}(G)} := \sup\limits_G |f| + |f|_{\Lambda^{k,\omega}(G)}$

- $S \mapsto E_k(S; f)$, $S \subset \mathbb{R}^n$: (local) best approximation of f by polynomials of degree $k-1$
- $\dot{\mathcal{E}}^{k,\omega}(S)$: the space of real functions on $S \subset \mathbb{R}^n$ equipped with the seminorm $f \mapsto \sup\left\{\dfrac{E_k(S \cap Q; f)}{\omega(r_Q)}\,;\, Q \in \mathcal{K}_S\right\}$

Spaces of differentiable and smooth functions

- $C^\ell(G)$: the space of ℓ-times continuously differentiable real functions on a domain G
- $\dot{C}_b^\ell(G)$: the subspace of $C^\ell(G)$ defined by finiteness of the seminorm

$$f \mapsto |f|_{C_b^\ell(G)} := \max_{|\alpha|=\ell} \sup_G |D^\alpha f|$$

- $\dot{C}_u^\ell(G)$: the subspace of $\dot{C}_b^\ell(G)$ consisting of functions with uniformly continuous higher derivatives
- $C_b^\ell(G)$: the subspace of $\dot{C}_b^\ell(G)$ defined by finiteness of the norm

$$f \mapsto \|f\|_{C_b^\ell(G)} := \sup_G |f| + |f|_{C_b^\ell(G)}$$

- $C^\ell\dot{\Lambda}^{k,\omega}(G)$: the subspace of $C^\ell(G)$ consisting of functions whose higher derivatives belong to $\dot{\Lambda}^{k,\omega}(G)$ equipped by the seminorm

$$f \mapsto |f|_{C^\ell\Lambda^{k,\omega}(G)} := \max_{|\alpha|=\ell} |D^\alpha f|_{\Lambda^{k,\omega}(G)}$$

- $C^\ell\Lambda^{k,\omega}(G)$: the subspace of $C^k\dot{\Lambda}^{k,\omega}(G)$ defined by the finiteness of the norm

$$f \mapsto \|f\|_{C^\ell\Lambda^{k,\omega}(G)} := \sup_G |f| + |f|_{C^\ell\Lambda^{k,\omega}(G)}$$

- $J^\ell\dot{\Lambda}^{k,\omega}(G)$ the space of ℓ-jets $\vec{f} := \{f_\alpha\}_{|\alpha|\le\ell}$ on G defined by finiteness of the seminorm

$$\vec{f} \mapsto |\vec{f}|_{J^\ell\Lambda^{k,\omega}(G)} := \max_{|\alpha|=\ell} |f_\alpha|_{\Lambda^{k,\omega}(G)}$$

- $J^\ell\Lambda^{k,\omega}(G)$: the subspace of $J^\ell\dot{\Lambda}^{k,\omega}(G)$ defined by finiteness of the norm

$$\vec{f} \mapsto ||f||_{J^\ell\lambda^{k,\omega}(G)} := \max_{|\alpha|\le\ell} |f_\alpha| + |\vec{f}|_{J^\ell\Lambda^{k,\omega}(G)}$$

Let X be one of the above introduced functions spaces on $\mathbb{R}^n$; let $S \subset \mathbb{R}^n$ be closed.

- $X|_S := \{f : S \to \mathbb{R}\,;\, f = g|_S$ for some $g \in X\}$: the trace of X to S equipped with the *trace seminorm*

$$f \mapsto |f|_{X|_S} := \inf\{|g|_X\,;\, g|_S = f\}$$

if X is seminormed and the analogous trace norm if X is normed

Extension constants

Let $\mathcal{M}, \mathcal{M}'$ be metric spaces and S be a metric subspace of $\mathcal{M}$.

- $L_{ext}(f)$: Lipschitz extension constant for $f : S \to \mathcal{M}'$ (the trace norm of f in $Lip(\mathcal{M}, \mathcal{M}')|_S$)
- $\Lambda(S, \mathcal{M}; \mathcal{M}') := \sup\left\{\dfrac{L_{ext}(f)}{L(f)}\,;\, f \in Lip(S, \mathcal{M}')\right\}$: (local) Lipschitz extension constant
- $\Lambda(\mathcal{M}, \mathcal{M}') := \sup\{\Lambda(S, \mathcal{M}, \mathcal{M}')\,;\, S \subset \mathcal{M}\}$: (global) Lipschitz extension constant

Let X be a Banach space.

- $Ext(S, X)$: the space of all bounded linear extension operators (*simultaneous extensions*) from $Lip(S, X)$ into $Lip(\mathcal{M}, X)$
- $\lambda(S, \mathcal{M}; X) := \inf\{\|E\|\,;\, E \in Ext(S, X)\}$: (local) linear Lipschitz extension constant
- $\lambda(\mathcal{M}, X) := \sup\{\lambda(S, \mathcal{M}; X)\,;\, S \subset \mathcal{M}\}$: (global) linear Lipschitz extension constant
- $\lambda(\mathcal{M}) := \lambda(\mathcal{M}, \mathbb{R})$

Let X be one of the above introduced spaces of differentiable or smooth functions on $\mathbb{R}^n$ (*smoothness spaces*) and Σ be a class of subsets in $\mathbb{R}^n$.

- $\delta_N(f; S; X) := \sup\{|f|_{X|_{S'}}\,;\, S' \subset S$ and $\operatorname{card} S' \le N\}$: (the seminorm here and below is replaced by the norm if X is normed)
- $\mathcal{F}_\Sigma(X)$: the *finiteness constant* of X with respect to Σ (the minimal N for which $X|_S$ coincides with linear space $\{g \in C(S)\,;\, \delta_N(g; S; X) < \infty\}$)
- $\mathcal{FP}(\Sigma)$: the class of all smoothness spaces with $\mathcal{F}_\Sigma(X) < \infty$
- $\gamma_\Sigma(X)$: the extension constant for X with respect to Σ equals

$$\sup\{|g|_{X|_S}\,;\, \delta_N(g; S; X) \le 1,\ S \in \Sigma\}$$

- $\mathcal{FP}_u(\Sigma)$: the subclass of $\mathcal{FP}(\Sigma)$ with $\gamma_\Sigma(X) < \infty$

For Σ being the class of all nonempty closed subsets the symbol Σ in all these notations is omitted.