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Linearization Models for Complex Dynamical Systems

Topics in Univalent Functions, Functional Equations and Semigroup Theory

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Preface

Interactions between Complex Analysis, Iteration Theory and Dynamical Systems have fascinated mathematicians for more than a century. Richness and beauty of these topics originated from the transparency of geometric ideas and depth of analytic methods forming a peculiarity of this area. Iteration algorithms and continuous dynamical processes are at the heart of many related fields, e.g., branching stochastic processes, composition operators, control theory and theory of linear operators in indefinite metric spaces. For recent books in these topics see, for example, [27, 104, 127, 130, 2, 40] and [120].

Each iteration procedure generates a discrete dynamical system, which can be considered a semigroup with respect to an integer parameter. The study of the behavior of such semigroups in a domain in the complex plane $\mathbb{C}$ goes back to the classical works of Julia, Fatou, Denjoy and Wolff and has been developed in different directions (including higher dimensional generalizations and dynamics in hyperbolic metric spaces and operator algebras, see [120] and references therein).

In parallel, and even earlier, the study of the local behavior of iterations near a fixed point was based on linearization models provided by the functional equations of Schröder, Abel and Bottcher. In 1884 Kœnigs showed how to solve Schröder's functional equation

$$h\left(F(z)\right) = \lambda h(z)$$

in a neighborhood of an attractive fixed point of F by using a limit scheme bearing his name.

Thus, the iteration properties of a given mapping F can be understood by means of the iteration properties of the linear mapping λw. This approach has been developed over a period of about a century by the combined efforts of many mathematicians (Valiron, Hadamard, Pommerenke, Baker and Cowen, among others). The key tool in these investigations is the ability to construct an intertwining mapping h for different types of holomorphic functions as well as the analysis of its uniqueness and geometric and analytic properties.

Although the theory of discrete dynamical systems was developed extensively, little was known about semigroups with respect to a continuous parameter. At the same time, continuous semigroups have various applications in Markov Stochastic

Processes [79, 126], Control Theory and Optimization [81]. For example, one of the central problems in Markov branching processes is to describe the asymptotic behavior of the process in a neighborhood of the so-called extinction probability which can be understood as the common attractive fixed point of a corresponding semigroup.

In 1978, Berkson and Porta [17] showed that each continuous semigroup $\{F_t\}_{t\geq 0}$ of holomorphic self-mappings on the open unit disk Δ is everywhere differentiable with respect to its parameter. Hence the limit

$$f(z) := \lim_{t\to 0^+} \frac{z - F_t(z)}{t} \tag{0.0.1}$$

exists and defines a holomorphic function f in Δ. This function is called the infinitesimal generator of the semigroup. Furthermore, it can be shown (see [17], [115] and [130]) that given f, the semigroup can be reproduced by the solution of the Cauchy problem:

$$\begin{cases} \dfrac{\partial F_t(z)}{\partial t} + f\left(F_t(z)\right) = 0, & t \geq 0, \\ F_0(z) = z, & z \in \Delta. \end{cases}$$

Moreover, in the same work Berkson and Porta showed that a holomorphic function f is a semigroup generator if and only if it admits the representation

$$f(z) = (z - \tau)(1 - z\overline{\tau})\, p(z), \quad z \in \Delta, \tag{0.0.2}$$

for some point $\tau \in \overline{\Delta}$ and some function $p \in \operatorname{Hol}(\Delta, \mathbb{C})$ with $\operatorname{Re} p(z) \geq 0$, $z \in \Delta$; and this representation is unique.

Based on this fact, Siskakis investigated in [136] continuous analogs of Schröder's and Abel's functional equations:

$$h(F_t(z)) = \lambda^t h(z) \tag{0.0.3}$$

and

$$h(F_t(z)) = h(z) + t. \tag{0.0.4}$$

It seems he was the first to focus on the connection between their solutions and the semigroup generators. In particular, he showed that if a one-parameter continuous semigroup has an interior fixed point, then Schröder's equation (0.0.3) (or Abel's equation (0.0.4) in the opposite case) should have the unique normalized solution.

The linearization models provided by these equations allowed Siskakis to study weighted semigroups of composition operators as well as the Cesàro and Volterra averaging operators on classical Hilbert and Banach spaces of analytic functions (Hardy spaces, Bergman spaces etc.).

Linearization models for continuous semigroups attracted special attention in the last decade. Recently, Contreras, Díaz-Madrigal and Pommerenke in a series of works [32, 33, 34] used these models to study dynamical properties of evolution equation and the boundary behavior of Kœnigs functions.

Also, we would like to emphasize that linearization models for discrete and continuous time dynamical systems are driving forces for the Composition Operator Theory on function spaces in the unit disk of the complex plane. Naturally, the eigenvalue problems for composition operators on Hardy, Bergman and Dirichlet spaces of analytic functions involve Schröder's functional equations. They can be considered linearization models for self-mappings of the underlying domain (see, for example, the books [127, 18] and [40]). In fact, as a part of operator theory, research into composition with a fixed function acting on a space of analytic functions is of recent origin, dating back to the mid-1960s, while investigation of continuous semigroups of composition operators started intensively from the work of Berkson and Porta at the beginning of the 1980s (see for example, the survey [136]).

Recent developments of the Denjoy–Wolff Theory and Julia–Carathéodory boundary versions of the Schwarz–Pick Lemma in the context of the generation theory for semigroups of holomorphic mappings lead to an independent study of linearization models for complex dynamical systems lying at the interface of analytic function theory and operator theory.

In addition, solutions of Schröder's and Abel's equations were applied to solve the Kœnigs embedding problem in [49] and the rigidity problem in the spirit of the Burns–Krantz theorem in [51, 53]. Different aspects of the boundary rigidity problem inspired by Abel's functional equation and its linearization model for holomorphic semigroups and generators can be integrated into the boundary interpolation theory for Schur's functions on the unit disk.

It turns out, that intertwining mappings for continuous semigroups have transparent geometric properties: they should be either starlike, spirallike, or convex in one-direction mappings. Thus these linearization models link to new questions in the classical Geometric Function Theory. Such questions have been investigated in [56, 5], as well as new covering and distortion theorems for spirallike and starlike functions studied in [47, 65, 132].

Finally, the nice geometry of linearization models enable the study of backward flow invariant domains for continuous semigroups [68, 69].

This book is devoted to a systematic and detailed survey and treatment of linearization models for one-parameter continuous semigroups, functional equations, different classes of univalent functions which serve intertwining mappings for these semigroups, and their applications to various problems of complex dynamics.

We believe that the book will be useful to a wide readership with an interest in these areas. Although many results have been previously published in journal papers, we endeavor to facilitate the understanding of their proofs to students and nonspecialists.

Some topics of this book have been given in special courses for students of ORT Braude College, Karmiel. More advanced issues were presented at the Seminars on Nonlinear Analysis at the Technion, Haifa, on Complex Analysis at the Bar–Ilan University, Ramat-Gan, on Operator Theory in the Banach Center of Polish Academy of Sciences, on Complex Analysis at the University Rome II Tor Vergata.

We would like to express our thanks to our colleagues who have read the manuscript, made useful remarks and suggested improvements. We are grateful to Professor Lawrence Zalcman who greatly influenced the conceptual approaches applied here. We also benefited from discussions with Professors Simeon Reich and Dov Aharonov. Many results included in the book were obtained due to collaboration with them.

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