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Homological Algebra of Semimodules and Semicontramodules

Semi-infinite Homological Algebra of Associative Algebraic Structures

Appendix C in collaboration with Dmitriy Rumynin
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To the memory of my father

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Preface*

The subject of this book is Semi-Infinite Algebra, or more specifically, Semi-Infinite Homological Algebra. The term “semi-infinite” is loosely associated with objects that can be viewed as extending in both a “positive” and a “negative” direction, with some natural position in between, perhaps defined up to a “finite” movement. Geometrically, this would mean an infinite-dimensional variety with a natural class of “semi-infinite” cycles or subvarieties, having always a finite codimension in each other, but infinite dimension and codimension in the whole variety [37]. (For further instances of semi-infinite mathematics see, e.g., [38] and [57], and references below.)

Examples of algebraic objects of the semi-infinite type range from certain infinite-dimensional Lie algebras to locally compact totally disconnected topological groups to ind-schemes of ind-infinite type to discrete valuation fields. From an abstract point of view, these are ind-pro-objects in various categories, often endowed with additional structures. One contribution we make in this monograph is the demonstration of another class of algebraic objects that should be thought of as “semi-infinite”, even though they do not at first glance look quite similar to the ones in the above list. These are *semialgebras* over coalgebras, or more generally over corings – *the* associative algebraic structures of semi-infinite nature.

The subject lies on the border of Homological Algebra with Representation Theory, and the introduction of semialgebras into it provides an additional link with the theory of corings [23], as the semialgebras are the natural objects dual to corings. The author’s main interests belong to Homological Algebra, and so the main body of the monograph consists of the formal development of the homological theory of corings and semialgebras, while the representation-theoretic (and other) examples and applications are relegated to appendices.

One such application worth mentioning here is related to the duality between complexes of representations of an infinite-dimensional Lie algebra with the complementary central charges, e.g., c and $26 - c$ for the Virasoro algebra [39, 77]. We interpret it as a particular case of a very general homological phenomenon related to coalgebras, which we call the *comodule-contramodule correspondence.* The latter is a coalgebra version of the Serre–Grothendieck duality – covariant, noncommutative, and not depending on any finiteness assumptions (the coalgebra $\mathcal{C}$ itself plays the role of the dualizing complex; cf. [65, 71]). This allows us to formulate the duality for infinite-dimensional Lie algebra representations as a (covariant) equivalence of triangulated categories.

On a less ambitious level, with the formal neighborhood of a closed subgroup in an algebraic group one can associate a semialgebra of (roughly speaking) dis-

*What follows is very speculative and should be taken with a grain of salt.

tributions on it, and the category of Harish-Chandra modules over an algebraic Harish-Chandra pair is the category of semimodules over this semialgebra. For further applications to Representation Theory, see [16], [17], and [45].

Another important area that Semi-Infinite Algebra and Geometry are related to is Mathematical Physics. The author of this monograph stands at the receiving end of a long chain of interpretative work through which the ideas originating in the interaction of Mathematics with Quantum Field Theory or String Theory are transferred to the heart of Algebra. We are not in a position to comment here on the possibilities of applications of the content of this book to Mathematical Physics, so we will restrict ourselves to a couple of references and some very general remarks. The semi-infinite homology of Lie algebras are closely related to what the physicists call the BRST construction [10, 11]; for a discussion of the significance of the semi-infinite homology in String Theory, see [41] and the introduction to [42].

The field of functions on the formal circle, which is the field of Laurent power series, is a very simple example of a semi-infinite algebraic object; and much more complicated algebraic or geometric objects built on the basis of the formal circle often have very visible semi-infinite structures. This includes the Virasoro and affine Kac-Moody Lie algebras, the varieties [58] and groups of formal loops, the semi-infinite flag variety [37], etc. The formal circle is obviously important for Conformal Field Theory [12], hence the significance of such objects of study as the semi-infinite homology [10, 40], the semi-infinite de Rham complex [58, 8], or the chiral differential operators [10, 59] in Mathematical Physics.

Things semi-infinite play a role in Class Field Theory [83] and the Langlands Program [11, 40] for the very same reason. A much more detailed discussion of the links between the semi-infinite cohomology and various other mathematical and physical disciplines can be found in the introduction to [84].

Another class of algebraic objects prominently featured in this monograph is that of *contramodules*. Their definition, introduced originally in the case of coalgebras or corings [33], can be extended to certain topological rings, topological Lie algebras, certain topological groups … These are modules with infinite summation operations, but of the kind that cannot be interpreted as any sort of limit with respect to a topology. Here one finds an approach to the infinite summation entirely different from the one most common to Analysis.

Typically, for an abelian category of "discrete", "smooth", or "torsion" modules there is an accompanying abelian category of contramodules. The latter contains all kinds of objects "dual" to the objects of the former, and some other objects in addition. For example, the category of "weakly l-complete abelian groups" appearing in the continuous étale cohomology theory [54] is simply the category of contramodules over the l-adic integers. While not "semi-infinite" in themselves, contramodules always come up whenever one wishes to pass from a semi-infinite *homology* to a semi-infinite *cohomology* theory.

One area where our approach is inspired by, still essentially different from, the classical one is Relative Homological Algebra. While the classical theory [47, 33, 80, 35] emphasizes relative derived functors of nonexact functors that may be quite conventional and not necessarily "relative" in themselves, here we are mainly interested in *absolute* derived functors, but the nonexact functors that we derive and the categories where they are defined are essentially relative by their nature. We always want our derived functors to assign long exact sequences of cohomology to arbitrary short exact sequences of complexes in the arguments, not only to short exact sequences that are split over some base. Still the base (or even two bases, one over the other) are built into the definitions of the categories and functors we work with.

One thing we cannot pretend to explain, still cannot avoid mentioning here, are the *exotic derived categories*. These are variations on the theme of the unbounded derived category. Their names are the (conventional unbounded) derived, the *coderived*, the *contraderived*, and the mixed, or *semiderived* categories. Historically, these first occurred in the derived nonhomogeneous Koszul duality theory [61], but from a wider point of view, the coderived and contraderived categories appear to be intrinsic to the comodules and contramodules (while occasionally useful for modules, too). For a definitive treatment of the exotic derived categories and their role in Koszul duality, we refer the reader to the long paper [76]. As to the nonhomogeneous Koszul duality itself, it is developed and used in this book as a strong technical tool.

An object of the contraderived category can be thought of as a complex having, in addition to the conventional cohomology at finite degrees, some kind of "cohomology in the degree $+\infty$". Analogously, a complex in the coderived category can be viewed as having a "cohomology in the degree $-\infty$". This is essential, in particular, for the construction of the comodule-contramodule correspondence, as the latter can well transform irreducible modules into acyclic complexes (i.e., those with no cohomology anywhere but "at infinity") and back. For example, an acyclic, but nontrivial object in the contraderived category of contramodules can be represented by an acyclic, unbounded complex of projective contramodules, and the latter thought of as a "left projective resolution of something living in the degree $+\infty$".

We also propose a very simple, bordering on self-evident, still apparently not widely known, approach to derived functors of two arguments, which allows us to obtain double-sided derived functors for free. It wouldn't get one too far without the exotic derived categories, though. The concrete double-sided derived functors we are interested in are the SemiExt and SemiTor over semiassociative semialgebras, and the semi-infinite (co)homology of Tate Lie algebras and locally compact totally disconnected topological groups. The semimodule-semicontramodule correspondence connects these with the more conventional one-sided Ext and CtrTor.

The functors to be derived are the *semitensor product* and *semihomomorphisms* in the semiassociative case, and the *semiinvariants* and *semicontrainvariants* in the Lie algebra or topological group case. These neither left, nor right exact functors are naturally associated with certain semi-infinite algebraic structures, and particularly with semialgebras. Still they are nontrivial enough even for finite-dimensional Lie algebras and finite groups.

To end these preparatory notes, let us say a few words about the state of the subject after this monograph. It appears that the question of *defining* the semi-infinite homology and cohomology generally, and in the case of associative algebraic structures specifically, has been now worked out and understood to a very significant extent. Compared to this development, our knowledge of the ways of *computing* the semi-infinite cohomology is next to nonexistent outside of the classical Lie algebra case. The only example where the semi-infinite cohomology of associative algebras has been computed as of now is that of the small quantum group with its triangular decomposition [4, 17]. The methods used for this computation have so far resisted, essentially, all attempts of transfer to other situations or generalization. Computing the semi-infinite cohomology remains a challenge for future researchers to take on.

It is our special pleasure to finish these most cursory remarks with a reference to B. Feigin's paper [36] that introduced the semi-infinite homology and the very term *semi-infinite*.

Introduction

This monograph grew out of the author's attempts to understand the definitions of semi-infinite (co)homology of associative algebras that had been proposed in the literature and particularly in the works of S. Arkhipov [2, 3] (see also [17, 79]). Roughly speaking, the semi-infinite cohomology is defined for a Lie or associative algebra-like object which is split in two halves; the semi-infinite cohomology has the features of a homology theory (left derived functor) along one half of the variables and a cohomology theory (right derived functor) along the other half.

In the Lie algebra case, the splitting in two halves only has to be chosen up to a finite-dimensional space; in particular, the homology of a finite-dimensional Lie algebra only differs from its cohomology by a shift of the homological degree and a twist of the module of coefficients. So one can define the semi-infinite homology of a Tate (locally linearly compact) Lie algebra [10] (see also [5]); it depends, to be precise, on the choice of a compact open vector subspace in the Lie algebra, but when the subspace changes it undergoes only a dimensional shift and a determinantal twist. For Lie superalgebras already, there is no such shift/twist phenomenon, and the dependence on the choice of a compact open subspace or subalgebra is very substantial [10]. Let us emphasize that what is often called the "semi-infinite cohomology" of Lie algebras should be thought of as their semi-infinite *homology*, from our point of view. What we call the semi-infinite *cohomology* of Tate Lie algebras is a different and dual functor, defined in this book (see Appendix D).

In the associative case, people usually considered an algebra A with two subalgebras N and B such that $N \otimes B \simeq A$ and there is a grading on A for which N is positively graded and locally finite-dimensional, while B is nonpositively graded. To this data, under certain assumptions, one assigns another graded algebra $A^{\#}$ with the same subalgebras N and B such that $B \otimes N \simeq A$. An attempt to understand this construction was the very starting point of the present research. We show that both the grading and the second subalgebra B are redundant; all one needs is an associative algebra R, a subalgebra K in R, and a coalgebra $\mathcal{C}$ dual to K. Certain flatness/projectivity and "integrability" conditions have to be imposed on this data. If they are satisfied, the tensor product $\boldsymbol{\mathcal{S}} = \mathcal{C} \otimes_K R$ has a *semialgebra* structure and all the machinery described below can be applied.

Furthermore, we propose the following general setting for semi-infinite (co)homology of associative algebraic structures. Let $\mathcal{C}$ be a coalgebra over a field k. Then $\mathcal{C}$-$\mathcal{C}$-bicomodules form a tensor category with respect to the operation of cotensor product over $\mathcal{C}$; the categories of left and right $\mathcal{C}$-comodules are module categories over this tensor category. Let $\boldsymbol{\mathcal{S}}$ be a ring object in this tensor category; we call such an object a *semialgebra* over $\mathcal{C}$ (due to it being "an algebra in half of the variables and a coalgebra in the other half"). One can consider module objects over $\boldsymbol{\mathcal{S}}$ in the module categories of left and right

$\mathcal{C}$-comodules; these are called left and right $\mathcal{S}$-*semimodules.* The categories of left and right semimodules are abelian if $\mathcal{S}$ is an injective right and left $\mathcal{C}$-comodule, respectively; let us suppose that it is. There is a natural operation of *semitensor product* of a right semimodule and a left semimodule over $\mathcal{S}$, denoted by

$$\Diamond_{\mathcal{S}}\colon \mathsf{simod}\text{-}\mathcal{S} \times \mathcal{S}\text{-}\mathsf{simod} \longrightarrow k\text{-}\mathsf{vect};$$

it can be thought of as a mixture of the cotensor product $\square_{\mathcal{C}}$ over $\mathcal{C}$ and the tensor product in the direction of $\mathcal{S}$ relative to $\mathcal{C}$. This functor is neither left, nor right exact. Its double-sided derived functor SemiTor is suggested as the associative version of semi-infinite *homology* theory.

Before describing the functor SemiHom (whose derived functor SemiExt provides the associative version of semi-infinite *cohomology*), let us discuss a little bit of abstract nonsense. Let E be an (associative, but noncommutative) tensor category, M be a left module category over it, N be a right module category, and K be a category such that there is a pairing between the module categories M and N over E taking values in K. This means that there are multiplication functors

$$\mathsf{E}\times\mathsf{E}\longrightarrow\mathsf{E}, \quad \mathsf{E}\times\mathsf{M}\longrightarrow\mathsf{M}, \quad \mathsf{N}\times\mathsf{E}\longrightarrow\mathsf{N}, \quad \text{and} \quad \mathsf{N}\times\mathsf{M}\longrightarrow\mathsf{K}$$

and associativity constraints for ternary multiplications $\mathsf{E}\times\mathsf{E}\times\mathsf{E}\to\mathsf{E}$, $\mathsf{E}\times\mathsf{E}\times\mathsf{M}\to\mathsf{M}$, $\mathsf{N}\times\mathsf{E}\times\mathsf{E}\to\mathsf{N}$, and $\mathsf{N}\times\mathsf{E}\times\mathsf{M}\to\mathsf{K}$ satisfying the appropriate pentagonal diagram equations. Let A be a ring object in E. Then one can consider the category ${}_A\mathsf{E}_A$ of A-A-bimodules in E, the category ${}_A\mathsf{M}$ of left A-modules in M, and the category N_A of right A-modules in N. If the categories E, M, N, and K are abelian, there are functors of tensor product over A, making ${}_A\mathsf{E}_A$ into a tensor category, ${}_A\mathsf{M}$ and N_A into left and right module categories over ${}_A\mathsf{E}_A$, and providing a pairing

$$\mathsf{N}_A\times{}_A\mathsf{M}\longrightarrow\mathsf{K}.$$

These new tensor structures are associative whenever the original multiplication functors were right exact.

Suppose that we want to iterate this construction, considering a coring object C in ${}_A\mathsf{E}_A$, the categories of C-C-bicomodules in ${}_A\mathsf{E}_A$ and C-comodules in ${}_A\mathsf{M}$ and N_A, etc. Since the functors of tensor product over A are not left exact in general, the cotensor products over C will be only associative under certain (co)flatness conditions. If one makes the next step and considers a ring object S in the category of C-C-bicomodules in ${}_A\mathsf{E}_A$, one discovers that the functors of tensor products over S are only partially defined. Considering partially defined tensor structures, one can indeed build this tower of module-comodule categories and tensor-cotensor products in them as high as one wishes. In this book, we restrict ourselves to 3-story towers of *semialgebras* over *corings* over (ordinary) rings, mainly because we don't know how to define unbounded (co)derived categories of (co)modules for any higher levels (see below).

Now let us introduce *contramodules.* The functor $(V,W) \longmapsto \mathrm{Hom}_k(V,W)$ makes the category opposite to the category of vector spaces into a module category over the tensor category of vector spaces. A contramodule over an algebra R or a coalgebra $\mathcal{C}$ is an object of the category opposite to the category of modules or comodules in $k\text{-}\mathsf{vect}^{\mathsf{op}}$ over the ring object R or the coring object $\mathcal{C}$ in $k\text{-}\mathsf{vect}$. One can easily see that an R-contramodule is just an R-module, while the vector space of k-linear maps from a $\mathcal{C}$-comodule to a k-vector space provides a typical example of $\mathcal{C}$-contramodule. Setting $\mathsf{E}=\mathsf{M}=k\text{-}\mathsf{vect}$ and $\mathsf{N}=\mathsf{K}=k\text{-}\mathsf{vect}^{\mathsf{op}}$ in the above construction, one obtains a right module category $\mathcal{C}\text{-}\mathsf{contra}^{\mathsf{op}}$ over the tensor category $\mathcal{C}\text{-}\mathsf{comod}\text{-}\mathcal{C}$ together with a pairing $\mathrm{Cohom}_{\mathcal{C}}^{\mathsf{op}}\colon \mathcal{C}\text{-}\mathsf{comod}\times\mathcal{C}\text{-}\mathsf{contra}^{\mathsf{op}}\longrightarrow k\text{-}\mathsf{vect}^{\mathsf{op}}$. Given a semialgebra $\mathcal{S}$ over $\mathcal{C}$, one can apply the construction again and obtain the category of $\mathcal{S}$-*semicontramodules* and the functor

$$\mathrm{SemiHom}_{\mathcal{S}}^{\mathsf{op}}\colon \mathcal{S}\text{-}\mathsf{simod}\times\mathcal{S}\text{-}\mathsf{sicntr}^{\mathsf{op}}\longrightarrow k\text{-}\mathsf{vect}^{\mathsf{op}}$$

assigning a vector space to an $\mathcal{S}$-semimodule and an $\mathcal{S}$-semicontramodule. Though comodules and contramodules are quite different, there is a strong duality-analogy between them on the one hand, and an equivalence of their appropriately defined (exotic) unbounded derived categories on the other hand (see below).

Let us explain how we define double-sided derived functors. While the author knows of no natural way to define a derived functor of one argument that would not be either a left or a right derived functor, such a definition of derived functor *of two arguments* does exist in the balanced case. Namely, let $\Theta\colon \mathsf{H}_1\times\mathsf{H}_2\longrightarrow\mathsf{K}$ be a functor and $\mathsf{S}_i\subset\mathsf{H}_i$ be localizing classes of morphisms in categories H_1 and H_2. We would like to define a derived functor

$$\mathbb{D}\Theta\colon \mathsf{H}_1[\mathsf{S}_1^{-1}]\times\mathsf{H}_2[\mathsf{S}_2^{-1}]\longrightarrow\mathsf{K}.$$

Let F_1 be the full subcategory of "flat objects in H_1 relative to Θ" consisting of all objects $F\in\mathsf{H}_1$ such that the morphism $\Theta(F,s)$ is an isomorphism in K for any morphism $s\in\mathsf{S}_2$. Let F_2 be the full subcategory in H_2 defined in the analogous way. Suppose that the natural functors

$$\mathsf{F}_i[(\mathsf{S}_i\cap\mathsf{F}_i)^{-1}]\longrightarrow\mathsf{H}_i[\mathsf{S}_i^{-1}]$$

are equivalences of categories. Then the restriction of the functor Θ to the subcategory $\mathsf{F}_1\times\mathsf{H}_2$ of the Cartesian product $\mathsf{H}_1\times\mathsf{H}_2$ factorizes through $\mathsf{F}_1[(\mathsf{S}_1\cap\mathsf{F}_1)^{-1}]\times\mathsf{H}_2[\mathsf{S}_2^{-1}]$ and therefore defines a functor on the category $\mathsf{H}_1[\mathsf{S}_1^{-1}]\times\mathsf{H}_2[\mathsf{S}_2^{-1}]$. The same derived functor can be obtained by restricting the functor Θ to the subcategory $\mathsf{H}_1\times\mathsf{F}_2$ of $\mathsf{H}_1\times\mathsf{H}_2$. This construction can be even extended to partially defined functors of two arguments Θ (see 2.7).

For this definition of the double-sided derived functor to work properly, the localizing classes in the homotopy categories have to be carefully chosen (see 0.2.3). That is why our derived functors SemiTor and SemiExt are not defined on the conventional derived categories of semimodules and semicontramodules, but on their *semiderived categories.* The semiderived category of $\mathcal{S}$-semi(contra)modules is a mixture of the usual derived category in the module direction (relative to $\mathcal{C}$) and the *co/contraderived* category in the $\mathcal{C}$-co/contramodule direction. More precisely, one defines the semiderived category of $\mathcal{S}$-semimodules as the quotient category of the homotopy category of complexes of $\mathcal{S}$-semimodules by the thick subcategory formed by those complexes that, *considered as complexes of* $\mathcal{C}$*-comodules, vanish in the coderived category of* $\mathcal{C}$*-comodules.* The coderived category of $\mathcal{C}$-comodules is equivalent to the homotopy category of complexes of injective $\mathcal{C}$-comodules, and analogously, the contraderived category of $\mathcal{C}$-contramodules is equivalent to the homotopy category of complexes of projective $\mathcal{C}$-contramodules. So the distinction between the derived and co/contra/semiderived categories is only relevant for unbounded complexes and only in the case of infinite homological dimension.

A notable attempt to develop a general theory of semi-infinite homological algebra was undertaken by A. Voronov in [84]. Let us point out the differences between our approaches. First of all, Voronov only considers the semi-infinite homology of Lie algebras, while we work with associative algebraic structures. Secondly, Voronov constructs a double-sided derived functor of a functor of one argument and the choice of a class of resolutions becomes an additional ingredient of his construction, while we define double-sided derived functors of functors of two arguments and the conditions imposed on resolutions are determined by the functors themselves. Thirdly, Voronov works with graded Lie algebras and his functor of semivariants is obtained as the image of the invariants with respect to one half of the Lie algebra in the coinvariants with respect to the other half, while we consider ungraded Tate Lie algebras with only one subalgebra chosen, and our functor of *semiinvariants* is constructed in a much more delicate way (see below). Finally, no exotic derived categories appear in [84].

Another approach to the semi-infinite cohomology (but not homology) was developed in [17]. The definition of the semi-infinite cohomology of finite-dimensional associative algebras proposed in [17] agrees with the one given in this book when the algebra has a grading satisfying the restrictive conditions under which the main argument of [17] applies, but is not readily comparable to our definition in the general case. Indeed, for finite-dimensional algebras and modules the object $\operatorname{SemiExt}_{\mathcal{S}}(\mathfrak{M}, \mathfrak{P})$, being dual to the object $\operatorname{SemiTor}^{\mathcal{S}}(\mathfrak{P}^*, \mathfrak{M})$, is a complex of profinite-dimensional (compact) vector spaces; while the semi-infinite cohomology as defined in [17] are discrete vector spaces. However, there is a natural map from the latter to the former.

The coderived category of $\mathcal{C}$-comodules and the contraderived category of $\mathcal{C}$-contramodules turn out to be naturally equivalent. This equivalence can be thought of as a covariant analogue of the contravariant functor $\mathbb{R}\operatorname{Hom}_R(-, R)\colon \mathsf{D}(R\text{-}\mathsf{mod}) \longrightarrow \mathsf{D}(\mathsf{mod}\text{-}R)$ on the derived category of modules over a ring R. Moreover, there is a natural equivalence between the semiderived categories of $\mathcal{S}$-semimodules and $\mathcal{S}$-semicontramodules. The functors $\mathbb{R}\Psi_{\mathcal{S}}\colon \mathsf{D}^{\mathsf{si}}(\mathcal{S}\text{-}\mathsf{simod}) \longrightarrow \mathsf{D}^{\mathsf{si}}(\mathcal{S}\text{-}\mathsf{sicntr})$ and $\mathbb{L}\Phi_{\mathcal{S}}\colon \mathsf{D}^{\mathsf{si}}(\mathcal{S}\text{-}\mathsf{sicntr}) \longrightarrow \mathsf{D}^{\mathsf{si}}(\mathcal{S}\text{-}\mathsf{simod})$ providing this equivalence are defined in terms of the spaces of homomorphisms in the category of $\mathcal{S}$-semimodules and the operation of *contratensor product* of an $\mathcal{S}$-semimodule and an $\mathcal{S}$-semicontramodule. The latter is a right exact functor which resembles the functor of tensor product of modules over a ring. This equivalence of triangulated categories transforms the functor $\mathrm{SemiExt}_{\mathcal{S}}$ into the functors Ext in either of the semiderived categories (and the functor $\mathrm{SemiTor}^{\mathcal{S}}$ into the left derived functor $\mathrm{CtrTor}^{\mathcal{S}}$ of the functor of contratensor product). We call this kind of equivalence of triangulated categories the *comodule-contramodule correspondence* or the *semimodule-semicontramodule correspondence.*

The duality-analogy between semimodules and semicontramodules partly breaks down when one passes from homological algebra to the structure theory. Comodules over a coalgebra over a field are simplistic creatures; contramodules are quite a bit more complicated, though still much simpler than modules over a ring, the structure theory of a coalgebra over a field being much simpler than that of an algebra or a ring. We construct some relevant counterexamples. There is an analogue of Nakayama's Lemma for contramodules, a description of contramodules over an infinite direct sum of coalgebras, etc. These results can be extended to contramodules over certain topological rings (much more general than the topological algebras dual to coalgebras). Contramodules over topological Lie algebras can also be defined; and an isomorphism of the categories of contramodules over a topological Lie algebra and its topological enveloping algebra can be proven under certain assumptions.

A *coring* $\mathcal{C}$ over a ring A is a coring object in the tensor category of bimodules over A. (In a different terminology, this is called a *coalgebroid*.) A *semialgebra* $\mathcal{S}$ over a coring $\mathcal{C}$ is a ring object in the tensor category of bicomodules over $\mathcal{C}$; for this definition to make sense, certain (co)flatness conditions have to be imposed on $\mathcal{C}$ and $\mathcal{S}$ to make the cotensor product of bicomodules well-defined and associative. Throughout this monograph (with the exception of Chapter 0 and the appendices) we work with corings $\mathcal{C}$ over noncommutative rings A and semialgebras $\mathcal{S}$ over $\mathcal{C}$. Mostly we have to assume that the ring A has a finite homological dimension – for a number of reasons, the most important one being that otherwise we don't know how to define appropriately the unbounded (co)derived category of $\mathcal{C}$-comodules. No assumptions about the homological dimension of the coring and the semialgebra are made. Besides, we mostly have to suppose that $\mathcal{C}$ is a flat left and right A-module and $\mathcal{S}$ is a coflat left and right $\mathcal{C}$-comodule, and even certain (co)projectivity conditions have to be imposed in order to work with contramodules.

All kinds of relative adjustness (flatness, projectivity, injectivity) properties are considered in this monograph, but their definitions differ from the ones typical of the classical relative homological algebra in one important respect. Specifically, we define relative adjustness in terms of complexes or exact triples *adjusted* over the base, rather than those *split* over the base. Our relative conditions tend to be weaker than the ones defined in the classical manner, more delicate, and better behaved with respect to the exact sequences that do not split over the base. When we need to consider relative adjustness properties defined in ways more resembling the classical approach, we insert the word "quite" into our terms for such properties.

Nonhomogeneous quadratic duality [73, 75] establishes a correspondence between nonhomogeneous Koszul algebras and Koszul CDG-algebras. This duality has a relative version with a base ring, assigning, e.g., the de Rham DG-algebra to the filtered algebra of differential operators (the base ring being the ring of functions, in this case). For a number of reasons, it is advisable to avoid passing to the dual vector space/module in this construction, working with CDG-coalgebras instead of CDG-algebras; in particular, this allows one to include infinitely (co)generated Koszul algebras and coalgebras [74, 76]. In the relative case, this means considering the graded coring of polyvector fields, rather than the graded algebra of differential forms, as the dual object to the differential operators. The relevant additional structure on the polyvector fields (corresponding to the de Rham differential on the differential forms) is called a *quasi-differential structure*. Another important version of relative nonhomogeneous quadratic duality uses base coalgebras in place of base rings. This situation is simpler in some respects, since one still obtains CDG-coalgebras as the dual objects. As a generalization of these two dualities, one can consider nonhomogeneous Koszul semialgebras over corings and assign Koszul quasi-differential corings over corings to them. The Poincaré–Birkhoff–Witt theorem for Koszul semialgebras claims that this correspondence is an equivalence of categories.

The relative nonhomogeneous Koszul duality theorem provides an equivalence between the semiderived category of semimodules over a nonhomogeneous Koszul semialgebra and the coderived category of quasi-differential comodules over the corresponding quasi-differential coring, and an analogous equivalence between the semiderived category of semicontramodules and the contraderived category of quasi-differential contramodules. In particular, for a smooth algebraic variety M and a vector bundle E over M with a global connection ∇, there is an equivalence between the derived category of modules over the algebra/sheaf of differential operators on M acting in the sections of E and the coderived category (and also the contraderived category, when M is affine) of CDG-modules over the CDG-algebra $\Omega(M, \operatorname{End}(E))$ of differential forms with coefficients in the vector bundle $\operatorname{End}(E)$. The differential d in $\Omega(M, \operatorname{End}(E))$ is the de Rham differential depending on ∇ and the curvature element $h \in \Omega^2(M, \operatorname{End}(E))$ is the curvature of ∇.

Natural examples of semialgebras and semimodules come from Lie theory. Namely, let $(\mathfrak{g}, H)$ be an algebraic Harish-Chandra pair, i.e., $\mathfrak{g}$ is a Lie algebra over a field k and H is a smooth affine algebraic group over k corresponding to a finite-dimensional Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$. Let $\mathcal{C}(H)$ be the coalgebra of functions on H. Then the category $\mathsf{O}(\mathfrak{g}, H)$ of Harish-Chandra modules is isomorphic to the category of left semimodules over the semialgebra $\mathcal{S}(\mathfrak{g}, H) = U(\mathfrak{g}) \otimes_{U(\mathfrak{h})} \mathcal{C}(H)$. If the group H is unimodular, the semialgebra $\mathcal{S} = \mathcal{S}(\mathfrak{g}, H)$ has an involutive anti-automorphism. In general, the opposite semialgebras $\mathbf{\mathcal{S}}$ and $\mathbf{\mathcal{S}}^{\mathrm{op}}$ are Morita-equivalent in some sense; more precisely, there is a canonical left $\mathbf{\mathcal{S}}\otimes_k\mathbf{\mathcal{S}}$-semimodule $\mathbf{\mathcal{E}} = \mathbf{\mathcal{E}}(\mathfrak{g}, H)$ such that the semitensor product with $\mathbf{\mathcal{E}}$ provides an equivalence between the categories of right and left $\mathbf{\mathcal{S}}$-semimodules. Geometrically, $\mathbf{\mathcal{E}}(\mathfrak{g}, H)$ is the bimodule of distributions on an algebraic group G supported in its subgroup H and regular along H. So the semitensor product of $\mathbf{\mathcal{S}}$-semimodules can be considered as a functor on the category $\mathsf{O}(\mathfrak{g}, H) \times \mathsf{O}(\mathfrak{g}, H)$. This functor factorizes through the functor of tensor product in the category $\mathsf{O}(\mathfrak{g}, H)$ and is closely related to the functor of $(\mathfrak{g}, H)$-*semiinvariants* $\mathbf{\mathcal{M}} \longmapsto \mathbf{\mathcal{M}}_{\mathfrak{g},H}$ on the category of $(\mathfrak{g}, H)$-modules. The semiinvariants are a mixture of invariants over H and coinvariants along $\mathfrak{g}/\mathfrak{h}$.

More generally, let $(\mathfrak{g}, H)$ be a *Tate Harish-Chandra pair*, that is $\mathfrak{g}$ is a Tate Lie algebra and H is an affine proalgebraic group corresponding to a compact open subalgebra $\mathfrak{h} \subset \mathfrak{g}$. Let $\varkappa\colon (\mathfrak{g}', H) \longrightarrow (\mathfrak{g}, H)$ be a morphism of Tate Harish-Chandra pairs with the same proalgebraic group H such that the Lie algebra map $\mathfrak{g}' \longrightarrow \mathfrak{g}$ is a central extension whose kernel is identified with k; assume also that H acts trivially in $k \subset \mathfrak{g}'$. One example of such a central extension of Tate Harish-Chandra pairs comes from the canonical central extension $\mathfrak{g}^\sim$ of $\mathfrak{g}$; we denote the corresponding morphism by $\varkappa_0$. There is a semialgebra

$$\mathbf{\mathcal{S}}_\varkappa(\mathfrak{g}, H) = U_\varkappa(\mathfrak{g}) \otimes_{U(\mathfrak{h})} \mathcal{C}(H)$$

over the coalgebra $\mathcal{C}(H)$ such that the category of left semimodules over $\mathbf{\mathcal{S}}_\varkappa = \mathbf{\mathcal{S}}_\varkappa(\mathfrak{g}, H)$ is isomorphic to the category of discrete $(\mathfrak{g}', H)$-modules where the unit central element of $\mathfrak{g}'$ acts by the identity (Harish-Chandra modules with the central charge $\varkappa$). Left semicontramodules over the opposite semialgebra $\mathbf{\mathcal{S}}_\varkappa^{\mathrm{op}}$ can be described in terms of compatible structures of $\mathfrak{g}'$-contramodules and $\mathcal{C}(H)$-contramodules. These are called *Harish-Chandra contramodules* with the central charge $-\varkappa$; the dual vector spaces to Harish-Chandra modules with the central charge $\varkappa$ can be found among them.

The semialgebras $\mathbf{\mathcal{S}}_\varkappa$ and $\mathbf{\mathcal{S}}^{\mathrm{op}}_{-\varkappa_0-\varkappa}$ are naturally isomorphic, at least, when the pairing $U(\mathfrak{h})\otimes_k \mathcal{C}(H) \longrightarrow k$ is nondegenerate in $\mathcal{C}(H)$. In view of the semimodule-semicontramodule correspondence theorem, it follows that the semiderived categories of Harish-Chandra modules with the central charge $\varkappa$ and Harish-Chandra contramodules with the central charge $\varkappa+\varkappa_0$ over $(\mathfrak{g}, H)$ are naturally equivalent. So the well-known phenomenon of correspondence between complexes of modules with complementary central charges over certain infinite-dimensional Lie algebras can be formulated as an equivalence of triangulated categories using the notions of contramodules and semiderived categories. Besides, it follows that the category of right semimodules over $\mathbf{\mathcal{S}}_\varkappa$ is isomorphic to the category of Harish-Chandra modules with the central charge $-\varkappa - \varkappa_0$. When the proalgebraic group H is prounipotent (and $\mathfrak{h}$ is exactly the Lie algebra of H), the object

$$\mathrm{SemiTor}^{\mathbf{\mathcal{S}}_\varkappa}(\mathbf{\mathcal{N}}^\bullet, \mathbf{\mathcal{M}}^\bullet)$$

of the derived category of k-vector spaces is represented by the complex of semi-infinite forms over $\mathfrak{g}$ with coefficients in the $\mathfrak{g}^\sim$-module $\mathbf{\mathcal{N}}^\bullet \otimes_k \mathbf{\mathcal{M}}^\bullet$. This provides a comparison of our theory of SemiTor with the semi-infinite homology of Tate Lie algebras. The semi-infinite cohomology of Lie algebras, whose coefficients are contramodules over (the canonical central extensions of) Tate Lie algebras, is related to SemiExt in the analogous way.

To a topological group G with an open profinite subgroup H and a commutative ring k one can assign a semialgebra $\mathbf{S}_k(G,H)$ over the coring $\mathcal{C}_k(H)$ of k-valued locally constant functions on H such that the categories of left and right semimodules over $\mathbf{S}_k(G,H)$ are isomorphic to the category of smooth G-modules over k. So the category of semimodules over $\mathbf{S}_k(G,H)$ does not depend on H, neither does the category of semicontramodules over $\mathbf{S}_k(G,H)$; all the semialgebras $\mathbf{S}_k(G,H)$ with a fixed G and varying H are naturally Morita equivalent. The semiderived categories of semimodules and semicontramodules over $\mathbf{S}_k(G,H)$ do depend on H essentially, however, as do the functors SemiTor and SemiExt over $\mathbf{S}_k(G,H)$. These double-sided derived functors may be called the *semi-infinite (co)homology of a group with an open profinite subgroup*. The semi-infinite homology of topological groups is a mixture of the discrete group homology and the profinite group cohomology.

Examples of corings $\mathcal{C}$ over commutative rings A for which the left and the right actions of A in $\mathcal{C}$ are different come from the algebraic groupoids theory, and examples of semialgebras over such corings come from Lie theory of groupoids. Namely, let (M,H) be a smooth affine groupoid, i.e., M and H are smooth affine algebraic varieties, there are two smooth morphisms $s_H, t_H\colon H \rightrightarrows M$ of source and target, and there are unit, multiplication, and inverse element morphisms satisfying the usual groupoid axioms. Let $A = A(M)$ be the ring of functions on M and $\mathcal{C} = \mathcal{C}(H)$ be the ring of functions on H. Then $\mathcal{C}$ is a coring over A. Moreover, suppose that (M,H) is a closed subgroupoid of a groupoid (M,G). Let $\mathfrak{g}$ and $\mathfrak{h}$ be the Lie algebroids over the ring A corresponding to the groupoids (M,G) and (M,H), and let $U_A(\mathfrak{g})$ and $U_A(\mathfrak{h})$ be their enveloping algebras. Then there is a semialgebra $\mathbf{S} = \mathbf{S}_M(G,H) = U_A(\mathfrak{g})\otimes_{U_A(\mathfrak{h})}\mathcal{C}(H)$ over the coring $\mathcal{C}$ and a canonical left $\mathbf{S}\otimes_k\mathbf{S}$-semimodule $\mathbf{E} = \mathbf{E}_M(G,H)$ providing an equivalence between the categories of right and left $\mathbf{S}$-semimodules. The semimodule $\mathbf{E}$ consists of all distributions on G twisted with the line bundle $s_G^*(\Omega_M^{-1})\otimes t_G^*(\Omega_M^{-1})$, supported in H and regular along H (where Ω_M denotes the bundle of top forms on M).

Examples of corings over noncommutative rings come from Noncommutative Geometry [62, 63]. Noncommutative stacks are represented as quotients of noncommutative affine schemes corresponding to rings A by actions of corings $\mathcal{C}$ over A. The cotensor product of $\mathcal{C}$-comodules can be understood as the group of global sections of the tensor product of a right and a left sheaf over a noncommutative stack, while the tensor product of sheaves itself does not exist.

Notice that the roles of the ring and coring structures in our constructions are not symmetric; in particular, we have to consider conventional derived categories along the algebra variables and co/contraderived categories along the coalgebra variables. The cause of this difference is that the tensor product of modules commutes with the infinite direct sums, but not with the infinite products. This can be changed by passing to pro-objects; consequently one can still define versions of derived functors Cotor and Coext over a coring $\mathcal{C}$ without making any homological dimension assumptions at all by considering pro- and ind-modules (see Remarks 2.7 and 4.7). A problem remains to construct the comodule-contramodule correspondence without any homological dimension assumptions on the ring A. Here we only manage to weaken the finite homological dimension assumption to the Gorensteinness assumption.

Algebras/coalgebras over fields and semialgebras over coalgebras over fields are briefly discussed in Chapter 0. Semialgebras over corings and the functors of semitensor product over them are introduced in Chapter 1, and important constructions of flat comodules and coflat semimodules are presented there. The derived functor SemiTor is defined in Chapter 2. Contramodules over corings and semicontramodules over semialgebras are introduced in Chapter 3, and the derived functor SemiExt is defined in Chapter 4. Equivalence of exotic derived categories of comodules and contramodules is proven in Chapter 5; and the same for semimodules and semicontramodules is done in Chapter 6. Functors of change of ring and coring for the categories of comodules and contramodules are introduced in Chapter 7; functors of change of coring and semialgebra for the categories of semimodules and semicontramodules are constructed in Chapter 8. Closed model category structures on the categories of complexes of semimodules and semicontramodules are defined in Chapter 9. The construction of a semialgebra depending on three embedded rings and a coring dual to the middle ring is considered in Chapter 10. The Poincaré–Birkhoff–Witt theorem and the Koszul duality theorem for nonhomogeneous Koszul semialgebras are proven in Chapter 11. The basic structure theory of contramodules over a coalgebra over a field is developed in Appendix A. We compare our theory of SemiExt and SemiTor with Arkhipov's and Sevostyanov's semi-infinite Ext and Tor in Appendix B. Semialgebras corresponding to Harish-Chandra pairs and their Hopf algebra analogues are discussed in Appendix C. Tate Harish-Chandra pairs are considered in Appendix D, and the theorem of comparison with semi-infinite cohomology of Tate Lie algebras is proven there. Semialgebras corresponding to topological groups are discussed in Appendix E. Pairs of algebraic groupoids are considered in Appendix F.

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One terminological note: we will generally use the words *the homotopy category of* (an additive category A) and *the homotopy category of complexes of* (objects from A) as synonymous. Analogously, *the homotopy category of complexes* (with a particular property) *over* A is a full subcategory of the homotopy category of A.

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