Microlocal Analysis, Sharp Spectral Asymptotics and Applications IV

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Magnetic Schrödinger Operator 2

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Preface

The Problem of the *Spectral Asymptotics*, in particular the problem of the *Asymptotic Distribution of the Eigenvalues*, is one of the central problems in the *Spectral Theory of Partial Differential Operators*; moreover, it is very important for the *General Theory of Partial Differential Operators*.

I started working in this domain in 1979 after R. Seeley [1] justified a remainder estimate of the same order as the then hypothetical second term for the Laplacian in domains with boundary, and M. Shubin and B. M. Levitan suggested me to try to prove Weyl's conjecture. During the past almost 40 years I have not left the topic, although I had such intentions in 1985, when the methods I invented seemed to fail to provide the further progress and only a couple of not very exciting problems remained to be solved. However, at that time I made the step toward local semiclassical spectral asymptotics and rescaling, and new much wider horizons opened.

So I can say that this book is the result of 40 years of work in the Theory of Spectral Asymptotics and related domains of Microlocal Analysis and Mathematical Physics (I started analysis of *Propagation of singularities* (which plays the crucial role in my approach to the spectral asymptotics) in 1975).

This monograph consists of five volumes. In this Volume IV we study magnetic Schrödinger operator, in non-smooth settings, or in dimensions 4 and higher, and also to eigenvalue asymptotics for such operators.

Victor Ivrii,
Toronto, June 10, 2019.

Contents

Introduction

In this Volume we consider magnetic Schrödinger operator in dimensions 2, 3 and higher under rather low smoothness conditions, consider specially it in dimension 4 and, finally, consider functional-analytic arguments of Volume II applied to magnetic Schrödinger sand Dirac operators.

Part VIII. Non-smooth in dimensions 2, 3 and higher

Chapter 18. 2D- and 3D-magnetic Schrödinger Operator with Irregular Coefficients. Here we generalize Chapter 13 to the case when the coefficients are not very smooth. It leads us to the necessity to a microlocal canonical form, which, due to the lack of the smoothness, is not only incomplete but also the lowest computed term in which is modified.

Section 18.1 contains a preliminary analysis, *Sections 18.2–18.5* cover $d = 2$ (Case of weak magnetic field, Canonical form, Tauberian theory and Calculations and main theorems–respectively).

Sections 18.6–18.9 cover $d = 3$ (exactly in the same order as for $d = 2$).

Chapter 19. Multidimensional Magnetic Schrödinger Operator. I. Full-rank case. Here we consider $d \geq 4$, assuming that $\operatorname{rank}(F_{jk})(x) = d$ at each point x, where (F_{jk}) is a skew-symmetric *magnetic intensity matrix*, $F_{jk} = (\partial_k A_j - \partial_j A_k)$ [1]. This is the main case if $d = 2r$ is even. Then this matrix has eigenvalues $\pm i f_j(x)$ with $j = 1, \dots, r$ and $f_j(x) > 0$.

This chapter does not completely generalize Chapter 18 since smoothness conditions here are higher. We also need two *non-resonance* conditions:

$$(0.1)_{2,3} \qquad f_j(x) \neq f_k(x) \quad \forall j \neq k \qquad \text{and} \qquad f_j(x) \neq f_k(x) + f_l(x) \quad \forall j, k, l$$

at each point x (where the second condition covers both cases $k \neq l$ and $k = l$).

[1] For non-Euclidean metrics we need to lift one of two indices.

Section 19.1 contains a preliminary analysis, *Section 19.2* covers the case of the weak magnetic field.

In *Section 19.3* we reduce our operator to a microlocal canonical form, which again is incomplete but there are two obstacles: insufficient smoothness but also a lack of higher order non-resonance conditions since we do not impose them.

Sections 19.4–19.6 cover cases of the weak-to-moderate, moderate, moderate-to-strong, and strong magnetic field respectively.

In *Section 19.7* we consider the case of a simple violation of *non-degeneracy* assumption.

Chapter 20. Multidimensional Magnetic Schrödinger Operator. II. Non-full-rank Case. Here we consider $d \geq 4$, now assuming that $\operatorname{rank}(F_{jk})(x) = 2r < d$ at each point x. This is the main case if $d = 2r + 1$ is odd. Then this matrix has eigenvalues $\pm i f_j(x)$ with $j = 1, \dots, r$ and $f_j(x) > 0$ and eigenvalue 0 of the constant multiplicity $d - 2r$.

This chapter does not completely generalize Chapter 18 since smoothness conditions here are higher. We also need two *non-resonance* conditions $(0.1)_{2,3}$.

Section 20.1 contains a preliminary analysis, *Section 20.2* covers the case of the weak magnetic field.

In *Section 20.3* we reduce operator to a microlocal canonical form with the same caveats as in Section 19.3.

In *Sections 20.4* and *20.5* we derive remainder estimates in the cases of the moderate and strong magnetic field respectively; in *Sections 20.6* and *20.7* we calculate asymptotics and prove main theorems.

In *Section 20.8* we consider the cases of a simple violation of main assumptions.

Part IX. Magnetic Schrödinger Operator in Dimension 4

Chapter 21. 4D-Schrödinger Operator with a Strong Degenerating Magnetic Field. We start analysis of the generic operators in dimension 4. Then there are two magnetic intensities $f_1(x)$ and $f_2(x)$ and in the generic case they do not vanish simultaneously. We consider a vicinity of $\Sigma = \{x\colon f_1(x) = 0\}$ and in the generic case $\nabla f_1|_\Sigma \neq 0$. However it is not a simple lift of Chapter 14 because now $\nabla f_1|_\Sigma$ belongs to $\operatorname{Ker} F$ not in every point of Σ but only on submanifold $\Lambda \subset \Sigma$, $\dim \Lambda = 1$.

Section 21.1 contains a preliminary analysis, *Section 21.2* covers the case of the weak magnetic field.

Sections 21.3–21.5 cover the case of the stronger magnetic field: in *Section 21.3* we reduce operator to a microlocal canonical form, in *Section 21.4* we prove a remainder estimate and in *Section 21.5* we calculate asymptotics and prove main theorems.

Chapter 22. Generic $4D$-Schrödinger Operator with the Strong Magnetic Field. In this chapter we conclude our analysis of the generic operators in dimension 4. Now we assume that neither magnetic intensity vanishes but non-resonance conditions $(0.1)_{2,3}$ are violated. More precisely, in the generic case $(0.1)_2$ can be violated on manifold of codimension 2 and $(0.1)_3$ can be violated on manifold of codimension 1 and these manifolds do not intersect.

Sections 22.1 and *22.2* contain a preliminary analysis, *Section 22.3* covers the case of the weak magnetic field.

Sections 22.4–22.5 cover the case of stronger magnetic field: in *Section 22.4* we prove a remainder estimate and in *Section 22.5* we calculate asymptotics and prove main theorems.

In *Section 22.6* we consider a case of V, vanishing somewhere.

Part X. Eigenvalue Asymptotics for Schrödinger and Dirac Operators with the Strong Magnetic Field

Chapter 23. Eigenvalue Asymptotics. $2D$ case and Chapter 24. Eigenvalue Asymptotics. $3D$ case. These chapters are devoted to the distribution of spectra for the Schrödinger and Dirac operators with the strong magnetic field. The idea is exactly the same as in Chapter 11, but now we consider the case when we apply the results of Chapters 13, 17 and 19 after the partition-rescaling-multiplication procedure. So we now have two local effective parameters: h_{eff} and μ_{eff}, and on elements with $\mu_{\mathsf{eff}} \gg 1$ we refer to the results of Chapters 13 and 17 while on elements with $\mu_{\mathsf{eff}} \le \mathsf{const}$ we refer to the results of Chapters 4 and 7.

In *Sections 23.2* and *24.2* we consider the case of a fixed spectral parameter.

In *Sections 23.3* and *24.3* we are interested in asymptotics with the spectral parameter tending to $+\infty$.

In *Sections 23.4* and *24.4* we are interested in asymptotics with the spectral parameter tending to ± 0. In particular, we consider the Schrödinger and Dirac operators in $\mathbb{R}^d$ with constant metric coefficients, vanishing scalar potential and constant non-vanishing magnetic field.

(a) Schrödinger and Dirac operators in $\mathbb{R}^2$ with constant metric coefficients, vanishing scalar potential and constant non-vanishing magnetic field have pure point spectra of infinite multiplicity. If we perturb these operators by a scalar potential, tending to 0 at $-\infty$, then each of these eigenvalues of infinite multiplicity decomposes to a sequence of eigenvalues of finite multiplicities, tending to a point of the essential spectrum (which is an eigenvalue of the unperturbed operator) and we are interested in the asymptotic distribution of eigenvalues in each sequence.

We also consider generalizations to even-dimensional full-rank case, and for magnetic field either growing, or decaying at infinity, or stabilizing to positively homogeneous of order 0.

(b) For $d = 3$ we obtain asymptotics of eigenvalues near the boundary of the essential spectrum. If the potential decreases slowly enough, the results of Chapters 13 and 17 could be applied.

(c) For $d = 3$ we obtain asymptotics of eigenvalues near the boundary of the essential spectrum. If the potential decreases quickly enough, the results of Chapters 13 and 17 fail to apply so we use the theory of spectral asymptotics for operators with operator-valued symbols: if the magnetic field is directed along x_3 we introduce an auxiliary space $\mathbb{H} = \mathscr{L}^2(\mathbb{R}_{x_3})$ and consider x' as the main variable.

(d) We reload case (b) to obtain sharper asymptotics, using the theory of spectral asymptotics for operators with operator-valued symbols as in the case (c).

In *Sections 23.5* and *24.5* we consider asymptotics with respect to all three parametrs τ, μ, h.