Microlocal Analysis, Sharp Spectral Asymptotics and Applications III

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Magnetic Schrödinger Operator 1

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Preface

The Problem of the *Spectral Asymptotics*, in particular the problem of the *Asymptotic Distribution of the Eigenvalues*, is one of the central problems in the *Spectral Theory of Partial Differential Operators*; moreover, it is very important for the *General Theory of Partial Differential Operators*.

I started working in this domain in 1979 after R. Seeley [1] justified a remainder estimate of the same order as the then hypothetical second term for the Laplacian in domains with boundary, and M. Shubin and B. M. Levitan suggested me to try to prove Weyl's conjecture. During the past almost 40 years I have not left the topic, although I had such intentions in 1985, when the methods I invented seemed to fail to provide the further progress and only a couple of not very exciting problems remained to be solved. However, at that time I made the step toward local semiclassical spectral asymptotics and rescaling, and new much wider horizons opened.

So I can say that this book is the result of 40 years of work in the Theory of Spectral Asymptotics and related domains of Microlocal Analysis and Mathematical Physics (I started analysis of *Propagation of singularities* (which plays the crucial role in my approach to the spectral asymptotics) in 1975).

This monograph consists of five volumes. In this Volume IV we study magnetic Schrödinger operator, in non-smooth settings, or in dimensions 4 and higher, and also to eigenvalue asymptotics for such operators.

Victor Ivrii,
Toronto, June 10, 2019.

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Introduction

This Volume is entirely devoted to the study of the *magnetic Schrödinger operator* in dimensions 2 and 3[1),2)]

$$H = (-ih\nabla - \mu \boldsymbol{A}(x))^2 + V(x), \tag{0.40}$$

and also Schrödinger-Pauli operator[1)]

$$H = ((-ih\nabla - \mu \boldsymbol{A}(x)) \cdot \boldsymbol{\sigma})^2 + V(x), \tag{0.41}$$

and Dirac operators[1)]

$$H = ((-ih\nabla - \mu \boldsymbol{A}(x)) \cdot \boldsymbol{\sigma}) + \sigma_0 M + V(x) \tag{0.42}$$

and

$$H = ((-ih\nabla - \mu \boldsymbol{A}(x)) \cdot \boldsymbol{\sigma}) + V(x) \tag{0.43}$$

with a small semiclassical parameter h and large *magnetic intensity parameter* (a coupling constant), responsible for the interaction of the particle with the magnetic field, μ. Here $\boldsymbol{\sigma} = (\sigma_1, \dots, \sigma_d)$ where $\sigma_1, \dots, \sigma_d$ are Pauli matrices, $\sigma_0, \sigma_1, \dots, \sigma_d$ are Dirac matrices, $\boldsymbol{A}$ is *magnetic vector potential.* Here for odd d operator (0.43) is not a special case of (0.42) with $M = 0$ since the former exists for D multiple of $2^{(d-1)/2}$ and the latter for D multiple of $2^{(d-1)/2}$; D is a dimension of matrices.

In this Volume we consider the most interesting cases $d = 2, 3$ and also assume that $\boldsymbol{A}(x)$, $V(x)$ (and metrics g) are smooth.

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1) Actually we consider more general form.

2) From now we denote operator by H.

Part VI. Smooth Theory in Dimensions 2 and 3.

Chapter 13. Schrödinger Operator. Standard Theory. Here we consider the Schrödinger operator, assuming that the the unit ball under consideration is entirely contained in the domain X and that the intensity of the magnetic field nowhere vanishes in this ball.

In *Section 13.2* a preliminary analysis is done and, in particular, $e(x,x,\lambda)$ is calculated explicitly in the case when the metrics and the scalar potential are constant and the vector potential is linear (so the intensity of the magnetic field is also constant) and $X = \mathbb{R}^d$; these formulae are good approximations for the general case, which we consider. We prove the estimates $|\mathcal{E}_\psi(0)| \le ch^{-d}$ in the general case and $|\mathcal{E}_\psi(0)| \le \mu^{-s}$ for $\mu h \ge C_0$; hence for the Schrödinger operator (but neither for the Schrödinger-Pauli nor for the Dirac operators!) we assume in what follows that $\mu h \le c$.

In *Section 13.3* we reduce the operator in question microlocally to canonical forms which are really different in the cases $d=2$ and $d=3$. For $d=2$ we obtain, modulo $O(\mu^{-s})$, $\mu^{-1}h$-pseudodifferential operators with respect to $x'=x_1$. For $d=3$ we obtain, also modulo $O(\mu^{-s})$, $(\mu^{-1}h,h)$-pseudodifferential operators with respect to $x'=(x_1,x_2)$. In both cases x_d enters in these forms only through the harmonic oscillator.

Based on this reduction, in *Section 13.4* we analyze the propagation of singularities with respect to x' in the case of the *weak-to-moderate magnetic field* $\mu \le h^{\delta-1}$ and, under some restrictions, we prove that on the intervals $[-T,-T_0]$ and $[T_0,T]$ the distribution $\sigma_\psi(t)$ is negligible, where either $T_0 \asymp \mu^{-1}$ or $T_0 \asymp h^{1-\delta}$, while on the interval $[-2T_0, 2T_0]$ for $\sigma_\psi(t)$ the standard formulae of Section 4.3 hold where $T \asymp 1$ for $d=3$ and $T \asymp \mu$ or larger under *non-degeneracy* assumptions for $d=2$; in the latter case singularities propagate along *cyclotrons* which are circles of radii $\asymp \mu^{-1}$ which however drift with a speed also $\asymp \mu^{-1}$. For $d=3$ singularities propagate along helices of radii $\asymp \mu^{-1}$ winding around *magnetic lines*.

We always assume that there is an appropriate cutting near the energy level 0 with respect to the dual variable τ. Applying the Tauberian, method we obtain asymptotics for $\mathcal{E}_\psi(0)$ with the principal part $\asymp h^{-d}$ and the remainder term $O(h^{-2})$ for $d=3$ and $O(\mu^{-1}h^{-1})$ or even better for $d=2$. We also analyze less restrictive conditions in this and the following sections and obtain asymptotics in these cases with the same or worse remainder estimates. Thus, in this section *the canonical form is applied only for the study of the long-time propagation of singularities in order to extend the time* *interval and improve the remainder estimate in the Weyl-type asymptotics.*

Next, in *Section 13.5* the case of the *moderate-to-strong magnetic field* $h^{-\delta} \le \mu \le C_0 h^{-1}$ is considered. Then the remainder term $O(\mu^{-s})$ in the microlocal canonical form is negligible and, after decomposition with respect to Hermitian functions of $\sqrt{\mu h^{-1}} x_d$, we obtain families of $\mu^{-1}h$-pseudodifferential operators with respect to $x' = x_1$ for $d = 2$ and $(\mu^{-1}h, h)$-pseudodifferential operators with respect to $x' = (x_1, x_2)$ for $d = 3$. These operators depend on the number n of the Hermitian function. Applying the results of Chapter 4 to these families of operators, we obtain asymptotics for $\mathcal{E}_\psi(0)$ under certain conditions with the same remainder estimates as in Section 13.4. Thus, now *the canonical form is used fully.* These asymptotics are not completely effective (i.e., one can calculate some terms only "in principle" but not "in fact").

The frameworks of this section and those of Section 13.4 overlap and under all of them both asymptotics are valid. These asymptotics are of different types, but comparing them, we can eliminate "superficial terms", obtaining effective asymptotics for $\mu \le C_0 h^{-1}$. These two ideas—*Application of the canonical form for the analysis of propagation of singularities and the extension of the interval for* $\sigma_\psi(t)$ and *comparison of the asymptotics obtained by different methods*—are very useful and we use them heavily in many chapters of the book.

Next, in *Section 13.6* we consider the case of the *strong magnetic field* $\mu \ge \epsilon_0 h^{-1}$. Then instead of V we should impose the same conditions to $V_n^* = V + (2n+1)\mu F$ for the Schrödinger operator and to $V_n^* = V + 2n\mu F$ for the Schrödinger-Pauli operator (with $n =\in \mathbb{Z}^+$ for $d = 2$, $n = 0$ for $d = 3$) and derive asymptotics with the principal part $\asymp \mu h^{1-d}$ and remainder estimates $O(1)$ for $d = 2$, $O(\mu h^{-1})$ for $d = 3$.

Further, in the case $d = 2$ it is possible that $V_n^* \le -\epsilon\mu h$ and $V_{n+1}^* \ge \epsilon\mu h$ everywhere in the ball in question; then $\tau = 0$ belongs to the *spectral gap* and in this case remainder estimate is $O(\mu^{-s})$.

Furthermore, in *Section 13.7* we improve results of Sections 13.4–13.7 under assumptions about the long-time dynamics (along *drift lines* for $d = 2$ and *magnetic lines* for $d = 3$).

Some possible generalizations (weakening or even dropping some assumptions) are discussed in *Section 13.8.*

Chapter 14. 2D-Schrödinger Operator with Strong Degenerating Magnetic Field. In this chapter we consider the same operators in dimension 2 only albeit intensity of magnetic field F vanishes along lines with $\nabla F \ne 0$.

Section 14.1 is devoted to a preliminary analysis; in *Section 14.2* we study the classical dynamics which is rather non-trivial with short periodic trajectories appearing in $\mu^{-\frac{1}{2}}$-vicinity of these degeneration lines.

In *Section 14.3* we study related quantum dynamics and prove the remainder estimate $O(\mu^{-\frac{1}{2}}h)$. Then in *Section 14.4* we calculate asymptotics with the same magnitude of the principal part and with a correction term due to aforementioned short periodic trajectories. Further, in *Section 14.5* we formulate main theorems (all as $\mu \le h^{-1}$).

Finally, in *Sections 14.5* and *14.6* we analyze cases of the *strong magnetic field* $h^{-1} \le \mu \le h^{-2}$ and *superstrong magnetic field* $\mu \ge h^{-2}$ respectively.

Chapter 15. 2D-Schrödinger Operator with Strong Magnetic Field near Boundary. In this chapter we consider the same 2-dimensional operators as in Chapter 13 albeit near smooth boundary which causes a significant modification of the dynamics: while inner "particles" (aka *bulk electrons*) travel along the same cyclotrons as before, near-boundary "particles" (aka *edge electrons*) "jump" along the boundary with an average speed $\asymp 1$. Drifting, bulk particles may collide with the boundary and become edge particles and edge particles may leave the boundary and become bulk particles so the global dynamics could be really complicated.

We analyze different cases (weak, moderate, strong magnetic field) in *Sections 15.2–15.4* respectively. There is a significant difference between operators with Dirichlet and Neumann boundary conditions , especially in the case of the strong magnetic field.

Also the former spectral gaps could be (at least partially) filled with *edge energy levels* in which case we need to study their distribution as well. They are less dense than eigenvalues in the *spectral bands*.

As usual, the first *Section 15.1* is devoted to the preliminary analysis and the last *Section 15.5* to some generalizations.

Part VII. Smooth Theory in Dimensions 2 and 3 (continued).

Chapter 16. Magnetic Schrödinger Operator: Short Loops, Pointwise Spectral Asymptotics and Asymptotics of Dirac Energy. In this chapter we study the same operators as in Chapter 13 but under different angle: we are interested in the *pointwise spectral asymptotics*, i.e. asymptotics of $e(x,x,\lambda)$ without mollification with respect to x, and also asymptotics of the expression (0.26); the results needed in Chapters 25–28.

As we know, for pointwise spectral asymptotics the important role play short loops (recall that in contrast to periodic trajectory the loop returns to the same spatial point albeit from the different direction) and for operators in question there are plenty of them for $d=2$.

The first half of the chapter is devoted to the case $d=2$: in *Section 16.2* we analyze a toy-model when explicit construction is possible and derive formulae which we would like to prove in the general case. The short loops cause either correction term if magnetic field is not too strong or even completely different asymptotics if magnetic field is sufficiently strong.

Then in *Section 6.2* we derive pointwise asymptotics, and in *Sections 16.4* and *16.5* we derive asymptotics of the expression (0.26).

In *Sections 16.6–16.9* we do the same as $d=3$; then effects of short loops is mitigated by a "free" movement along magnetic lines.

Finally, in *Section 16.9* we derive estimate of

$$\iint \bigl(e(x,x,\tau)-h^{-d}\mathcal{N}_x(\tau)\bigr)\bigl(e(y,y,\tau)-h^{-d}\mathcal{N}_x(\tau)\bigr)\omega(x,y)\,dxdy, \tag{0.44}$$

where $h^{-d}\mathcal{N}_x(\tau)$ is a Weyl expression for $e(x,x,\tau)$ for $d=2,3$.

Chapter 17. Dirac Operator with Strong Magnetic Field. Here we consider operators (0.42) and (0.43) and our analysis is very similar to one of Chapter 13 but some significant differences and complications appear. *Sections 17.1–17.7* run in parallel with Sections 13.2–13.8.

We should mention that here as in Section 5.3 our results are proven uniformly with respect to $M\geq 0$.