Part II
Mathematical Finance

Overview and Notation

This second part of this monograph concerns mathematical models for markets of exchange-traded assets. The goal is to address questions such as valuation of derivative securities, hedging or more generally portfolio selection, and assessment of risks.

Chapter 8 covers some families of equity models as examples of mathematically tractable models which take stylised features of real data into account. The subsequent chapter introduces basic concepts of financial mathematics for traded assets, in particular the fundamental notion of *arbitrage*. Its consequences on valuation and hedging of derivatives are discussed in Chap. 11, regardless of any specific model.

Advanced models involving processes with jumps are typically *incomplete*, i.e. perfect hedging strategies do not exist and superhedging strategies are forbiddingly expensive. As a way out, Chap. 12 provides an approach how to hedge as efficiently as possible in the sense that the variance of the portfolio is minimised. The alternative and at the same time related concept of utility-based hedging and pricing is discussed in Chap. 13. The latter relies on portfolio optimisation, which is discussed in Chap. 10 because it is interesting in its own right.

Chapter 14 turns to arbitrage-free models for fixed-income markets. It is explained how major approaches to interest rate markets, namely the short-rate, Heath–Jarrow–Morton, forward process and rational market models extend to discontinuous processes. As in Chap. 8 the focus is on tractable and at the same time flexible families.

We generally use the notation of Part I. In most chapters the notion of semimartingale characteristics is used. Given an $\mathbb{R}^d$-valued semimartingale X, we generally denote its local characteristics relative to the "truncation" function $h(x) = x$ by (a^X, c^X, K^X) and its modified second characteristics by $\widetilde{c}^X$, cf. Definition 4.3, (2.9), and (4.31). This implies secretly that we assume X to be a special semimartingale whose characteristics are absolutely continuous, cf. the end of Sect. 4.1 and note that $b^{X,\mathrm{id}} = a^X$.

Given an $\mathbb{R}^{d+n}$-valued semimartingale (X, Y) we write $c^{X,Y}$ and $\widetilde{c}^{X,Y}$ for the $\mathbb{R}^{d\times n}$-valued predictable processes satisfying

$$c_{ij}^{X,Y} \bullet I = \langle X_i^c, Y_j^c \rangle, \quad i = 1, \ldots, d, \quad j = 1, \ldots, n$$

and

$$\widetilde{c}_{ij}^{X,Y} \bullet I = \langle X_i, Y_j \rangle, \quad i = 1, \ldots, d, \quad j = 1, \ldots, n,$$

in line with e.g. (3.7). This implies that the local characteristics of (X, Y) are of the form

$$\left(a^{(X,Y)}, c^{(X,Y)}, K^{(X,Y)}\right) = \left(\begin{pmatrix} a^X \\ a^Y \end{pmatrix}, \begin{pmatrix} c^X & c^{X,Y} \\ c^{Y,X} & c^Y \end{pmatrix}, K^{(X,Y)}\right) \tag{II.1}$$

and its modified second characteristics can be written as

$$\widetilde{c}^{(X,Y)} = \begin{pmatrix} \widetilde{c}^X & \widetilde{c}^{X,Y} \\ \widetilde{c}^{Y,X} & \widetilde{c}^Y \end{pmatrix}. \tag{II.2}$$

Finally, we write c^{-1} for the *Moore–Penrose pseudoinverse* of a matrix or matrix-valued process c, which is a particular matrix satisfying $cc^{-1}c = c$. It allows us to use the formulas of Part II in cases where the corresponding matrices fail to be invertible. One can show that c^{-1} is non-negative and symmetric if this holds for c.

As a side remark, recall that $K^X = 0$ and $\widetilde{c}^X = c^X$ for continuous processes X. Moreover, $b^{X,h}$ does not depend on the truncation function h and coincides with the growth rate a^X in this case. Finally, both the covariation and the predictable covariation are of the form $[X_i, X_j] = \langle X_i, X_j \rangle = c_{ij}^X \bullet I$ for continuous X.