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Mathematical Finance

Springer

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To Carl-Philipp, Friederike, Sophie and to Birgit, Dörte, Matthies

Preface

Mathematical finance provides a quantitative description of financial markets, more specifically markets for exchange-traded assets, using mostly dynamic stochastic models. It is used to tackle three basic issues.

- *Valuation of assets*
  What can reasonably be said about the price of a financial asset? As opposed to economic theory, mathematical finance focuses mainly on *relative* valuation of securities in comparison to other assets. This is particularly useful and in fact indispensable for derivative securities, which are by definition strongly linked to corresponding underlying quantities in the market.
- *Optimal or at least reasonable portfolio selection*
  How shall an investor choose her portfolio of liquid securities? Here, the focus is on *hedging*, i.e. on minimising the risk which arises, for example, from selling derivative contracts to customers.
- *Quantification of risk*
  The random nature of asset prices naturally involves the risk of losses. How can it be quantified reasonably?

Mathematical finance has grown into a field which is by far too broad to be covered in a single book. Markets, products and risks are diverse and so are the mathematical models and methods which they require.

The starting point and focal point of this present monograph is continuous-time stochastic processes allowing for jumps. Most textbooks on mathematical finance are limited to diffusion-type setups, which cannot easily account for abrupt price movements. Such changes, however, play an important role in real markets, which is why models with jumps have become an established tool in the statistics and mathematics of finance. Just as importantly, purely discontinuous processes lead to a much wider variety of, at the same time flexible and tractable, models. For example, their marginal laws are often known explicitly, which is typically not the case for diffusions.

Compared to the abundant literature on continuous models, such as [29, 78, 149, 187, 204, 223, 223, 279], and many more, there still seems to be a scarcity of textbooks allowing for processes with jumps. Notable exceptions are the monographs [60] and [38, 160, 276]. Other useful texts such as [143, 263], address more specific questions rather than general principles of financial mathematics.

Our goal is twofold:

- to give an account of general semimartingale theory, stochastic control and specific classes of processes to the extent needed for the applications in the second part
- to introduce basic concepts such as arbitrage theory, hedging, valuation principles, portfolio choice and term structure modelling

In a single monograph, we cannot give a comprehensive overview of stochastic models with and without jumps in mathematical finance. Rather, we provide an introduction to the basic building blocks and principles, helping the reader to understand the advanced research literature and to come up with concrete models and solutions in more specific situations.

The book is divided into two parts. Part I introduces the stochastic analysis of general semimartingales along with the basics of stochastic control theory. We do not cover the whole theory with complete proofs, which can be found in a number of excellent mathematical monographs. Rather, we focus on concepts and results that are needed to apply the theory to questions in mathematical finance. Proofs are mostly replaced by informal illustrations along with references to the literature. Nevertheless, we made an effort to provide mathematically rigorous definitions and theorems.

Part II turns to both advanced models and basic principles of mathematical finance. It differs in style from Part I in the sense that results are stated as engineering-style *rules* rather than precise mathematical theorems with all the technical assumptions. For example, we do not distinguish between local and true martingales, and questions of existence and uniqueness are swept under the rug. This is done deliberately in order to make basic concepts accessible to the mathematically less inclined reader who wants to apply advanced stochastic models in practice and also to the non-specialist who wants to get an overview of the general ideas before delving more thoroughly into the subject.

The theory of Parts I and II simplifies occasionally if one focuses on stochastic processes without jumps. Major changes are summarised in Sect. A.7 for the convenience of the reader. Mathematical finance in the broad sense has produced some insights, which may seem counterintuitive and hence surprising to the novice in the field. We collect links to such results in Sect. A.8. Otherwise, the appendix mostly contains mathematical tools that are needed in the main part on the text.

This book could not have appeared in the present form without the help of many people. An incomplete list includes Aleš Černý, Sören Christensen, Friedrich Hubalek, Simon Kolb, Paul Krühner, Matthias Lenga, Johannes Muhle-Karbe, Arnd Pauwels, and Richard Vierthauer with whom we had long discussions, which had an effect on the contents of the book. Funda Akyüz and Britta Ramthun-Kasan assisted with preparing the manuscript. Partial financial support through *DFG Sachbeihilfe 1682/4-1* is gratefully acknowledged. We also benefited from the environment provided by the Freiburg Institute for Advanced Studies (FRIAS). We thank Catriona Byrne and Marina Reizakis from Springer-Verlag for their interest, encouragement and patience.

Errors can hardly be avoided in a text of this size. Since they will be discovered only gradually, we refer to www.math.uni-kiel.de/finmath/~book for an updated list of corrections. On this page, you can also find the *Scilab* code that we have used to generate the figures and numerical examples. Of course, any comments and in particular hints to errors are welcome.

Freiburg im Breisgau, Germany — Ernst Eberlein
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