Part I
Stochastic Calculus

Overview

A large part of mathematics for finance is written in the language of *stochastic processes*, i.e. of random functions of time. The calculus of these processes is introduced in this first part of the monograph. Some of its concepts naturally generalise notions from ordinary calculus, others are intrinsically linked to their probabilistic nature.

Even though this book deals with continuous-time models, we devote the first chapter to stochastic calculus in discrete time. The results themselves will not be needed in the sequel but they help to understand the intuition behind the corresponding concepts in continuous time.

The remaining chapters generalise important notions from ordinary calculus to the random case. *Lévy processes* can be viewed as the stochastic counterparts of linear functions. They are of interest in their own right but they also appear as building blocks of more general classes of processes. In Chap. 3 we cover the theory of *stochastic integration*, which is indispensable for mathematical finance. By contrast, it seems less obvious whether and how differentiation can be transferred to the random case. In Chaps. 4 and 5 we discuss *semimartingale characteristics* and *infinitesimal generators* as two natural candidates for a stochastic "differentiation". If Lévy processes and semimartingale characteristics generalise linear functions and derivatives, respectively, *affine Markov processes* correspond to solutions of linear ordinary differential equations. They play an important role in finance because of their flexibility and analytical tractability. Finally, we introduce the basic concepts of *stochastic optimal control* in Chap. 7 because many questions in Mathematical Finance are explicitly or implicitly related to optimisation.

Informal differential notation and arguments are used occasionally in the physics literature and to some extent in finance as well. We mimic such reasoning here in a few so-called *physicist's corners*. While these remarks may be insightful to some readers, they could confuse others with a more formal mathematical background. In the latter case they can be skipped altogether because they are primarily meant to illustrate separately stated rigorous mathematical statements.