Harmonic Quasiconformal Mappings and Hyperbolic Type Metrics

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Preface

The goal of this book is to present a research area in Geometric Function Theory concerned with harmonic quasiconformal mappings and hyperbolic type metrics defined on planar and multidimensional domains. The classes of quasiconformal and quasiregular mappings are well-established areas of study in this field as these classes are natural and fruitful generalizations of the class of analytic functions in the planar case. Harmonic mappings are another natural generalization of conformal mappings and analytic functions and form another well-established class. Injective quasiregular mappings are quasiconformal, and conformal mappings are both harmonic and quasiconformal. On the other hand, harmonic mappings are smooth, and when quasiregular, they are also locally quasiconformal independently of the dimension. So in higher dimensions, the study of the class of mappings that are both harmonic and quasiconformal suggests itself. It turns out that while this seems at first a rather restrictive class, the study of this class uncovers new and unexpected phenomena and is today recognized as an important research area in Geometric Function Theory. The book contains many concrete examples, as well as detailed proofs and explanations of motivations behind given results, gradually bringing the reader to the forefront of current research in the area. The book is written for a wide readership from graduate students of mathematical analysis to researchers working in this or related areas who want to learn the tools or work on the open problems, many of which are listed in various parts of the book, especially in the last chapter. An extensive bibliography of the field is also given for the readers who wish to explore deeper into the results presented in the book or related results that are not covered here. Prerequisite knowledge for reading this book includes the basic knowledge of real and complex analysis, harmonic functions, and the topology of metric spaces. The book is primarily based on research done in the last 12 years, starting with the author's master and doctoral dissertations and followed by a number of papers that are either single authored or jointly authored with other experts in the field. The author is therefore grateful to all her collaborators and other mathematicians who have built this research area and have shared their expertise with enthusiasm. Without their help, this book would not have come into its existence.

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Vesna Todorčević

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Notation

$\mathbb{Z}$	Set of integers
$\mathbb{N}$	Set of positive integers
$\mathbb{R}$	Set of real numbers
$\mathbb{C}$	Set of complex numbers
$\|z\|$	Modulus of the complex number z
$\arg(z)$	Argument of the complex number z
$\mathbb{R}^n$	n-dimensional Euclidean space
$\overline{\mathbb{R}^n} = \mathbb{R}^n \cup \{\infty\}$	Möbius space
$\|x\|$	Euclidean norm of a vector $x \in \mathbb{R}^n$
$\mathbb{B}^n(x, r)$	Open ball centered at $x \in \mathbb{R}^n$ with radius $r > 0$
V_n	Volume of the n-dimensional unit ball
$S^{n-1}(x, r)$	Sphere centered at $x \in \mathbb{R}^n$ with radius $r > 0$
ω_{n-1}	$(n-1)$-dimensional measure of $S^{n-1}(0, 1)$
$\mathbb{H}^n$	Poincare half-space
S_ρ	Planar angular domain
$P(a, t)$	$(n-1)$-dimensional hyperplane
$\mathcal{GM}(\overline{\mathbb{R}^n})$	Group of Möbius transformations
$M_3(\mathbb{R})$	Set of square matrices of order 3
$\pi(x)$	Stereographic projection
$q(x, y)$	Spherical (chordal) distance between x and y
$Q(x, r)$	Spherical ball
$\|a, b, c, d\|$	(Absolute) cross ratio
a^*	Image of the point a under an inversion on S^{n-1}
$D(a, M)$	Hyperbolic ball with center a and radius M
$\mathrm{dist}(x, A)$	Distance of a point $x \in \mathbb{R}^n$ to a set $A \subseteq \mathbb{R}^n$
∂A	Boundary of a set $A \subseteq \mathbb{R}^n$
$diam(A)$	Diameter of a set $A \subseteq \mathbb{R}^n$
χ_A	Characteristic function of a set $A \subseteq \mathbb{R}^n$
$l(\gamma)$	Length of the curve γ
$\rho(x, y)$	Hyperbolic distance between x and y
j_D	Distance ratio metric

k_D	Quasihyperbolic metric
δ_G	Seittenranta metric
α_G	Apollonian metric
s_G	Triangular ratio metric
λ_G	Ferrand metric
μ_G	Modulus metric
$\Delta(E, F; G)$	Family of all closed nonconstant curves joining E, F in G
$R_{G,n}(s)$	Grötzch ring
$R_{T,n}(s)$	Teichmüller ring
$\gamma(s)$, $\gamma_n(s)$	Capacity of $R_{G,n}(s)$
$\tau(s)$, $\tau_n(s)$	Capacity of $R_{T,n}(s)$
$\varphi_K(r)$, $\varphi_{K,n}(r)$	Special function related to the Schwarz lemma
$\log \Phi_n(s)$	Modulus of the Grötzsch ring
$\log \psi_n(s)$	Modulus of the Teichmüller ring
λ_n	Grötzsch ring constant
$mod(R)$	Modulus of a ring
$cap(R)$	Capacity of a ring
$\mathrm{cap}_W(K)$	Wiener capacity
p-$cap\ E$, $cap\ E$	(p-)capacity of a condenser
$\Lambda_\alpha(F)$	α-dimensional Hausdorff measure of F
$N(f, A, y)$	Number of preimages of point y in A under f
$Lip(f)$	Lipschitz constant of f
$K(f)$, $K_O(f)$, $K_I(f)$	Maximal, outer, and inner dilatation of f
$H(x, f)$	Linear dilatation of a mapping f at x
$M_p(\Gamma)$, $M(\Gamma)$	(p-)modulus of the curve Γ
$Id_K(\partial\mathbb{B}n)$	K-qc maps with identity boundary values
$C^1(C^2)$	Class of functions with continuous first-order (second-order) derivatives
$C_C^\infty(\Omega)$	Class of compactly supported functions with derivatives of all orders
∇f	Gradient of mapping $f : \Omega \longrightarrow \mathbb{R}^n$
Df	Weak derivative of mapping f
$J_f(x)$	Jacobian of mapping $f : \Omega \longrightarrow \mathbb{R}^n$ at x
$\alpha_f(z)$	Average derivative
$\mathscr{H}_u$	Hessian of u
$\|\|Du(x)\|\|$	Hilbert–Schmidt norm
Δf	Laplacian of f
$L^p(\Omega)$	Lebesgue space
$\|\|f\|\|_{L^p}$	L^p-norm of function f
L^p_{loc}	Local Lebesgue space
$W^{1,p}(\Omega)$	Sobolev space
$W^{1,p}_{loc}(\Omega)$	Local Sobolev space
$P(x, \xi)$	Poisson kernel for the unit ball
$G_{\mathbb{B}^n}(x, y)$	Green function on the unit ball

$\mathscr{I}_s h$	Riesz potential of order s
$QNS_K(\Omega)$	Class of all quasi-nearly subharmonic functions for a fixed K
$QNS(\Omega)$	Class of all quasi-nearly subharmonic function defined in Ω
$RO(\Omega)$	Class of regularly oscillating functions in Ω
$\|\|u\|\|_{QNS}$	QNS-norm of function u
$\|\|u\|\|_{RO}$	RO-norm of function u
$\|\|u\|\|_{BMO}$	BMO-norm of function u

Preliminaries

The background material necessary for reading this book can be found in standard texts of this area of mathematics such as [23, 138] and [155]. We shall also try to follow the standard terminology and notation as much as possible, but in this chapter, we list the notation and the terminology that are specific for this book.

A pair (X, d) is called a metric space if $X \neq \emptyset$ and $d : X \times X \to [0, +\infty)$ satisfies the following four conditions:

(M1) $d(x, y) \geqslant 0$, for all $x, y \in X$,
(M2) $d(x, y) = 0$ iff $x = y$,
(M3) $d(x, y) = d(y, x)$ for all $x, y \in X$,
(M4) $d(x, z) \leqslant d(x, y) + d(y, z)$ for all $x, y, z \in X$.

A pair (X, d) is called a pseudometric space if $X \neq \emptyset$ and $d : X \times X \to [0, +\infty)$ satisfies the following four conditions:

(M1) $d(x, y) \geqslant 0$, for all $x, y \in X$,
(M2') $d(x, x) = 0$,
(M3) $d(x, y) = d(y, x)$ for all $x, y \in X$,
(M4) $d(x, z) \leqslant d(x, y) + d(y, z)$ for all $x, y, z \in X$.

The inner product $\langle (a_1, \ldots, a_n), (b_1, \ldots, b_n) \rangle$ is defined to be equal to the sum $a_1 b_1 + \cdots + a_n b_n$, and we let $|a| = \sqrt{\langle a, a \rangle}$.

Let (X, d_1) and (Y, d_2) be metric spaces, and let $f : X \to Y$ be a continuous mapping. Then, we say that f is uniformly continuous if there exists an increasing continuous function $\omega : [0, \infty) \to [0, \infty)$ with $\omega(0) = 0$ such that

$$d_2(f(x), f(y)) \leq \omega(d_1(x, y)) \text{ for all } x, y \in X.$$

We call the function ω the modulus of continuity of f. If there exist $C, \alpha > 0$ such that $\omega(t) \leq C t^{\alpha}$ for all $t \in (0, 1)$, we say that f is Hölder continuous with

Hölder exponent α. If $\alpha = 1$, we say that f is Lipschitz with the Lipschitz constant C or simply C-Lipschitz. If f is a homeomorphism and both f and f^{-1} are C-Lipschitz, then f is C-bi-Lipschitz or C-quasiisometry, and if $C = 1$, we say that f is an isometry. These conditions are said to hold locally if they hold for each compact subset of X.

A very special case of these mappings are the isometries. Recall that if (X_1, d_1) and (X_2, d_2) are metric spaces and $f : X_1 \to X_2$ a homeomorphism, then we call f an *isometry* if $d_2(f(x), f(y)) = d_1(x, y)$ for all $x, y \in X_1$.

We use the notation

$$B^n(x, r) = \{y \in \mathbb{R}^n : |x - y| < r\},$$

$$S^{n-1}(x, r) = \{y \in \mathbb{R}^n : |x - y| = r\},$$

$$\mathbb{H}^n = \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_n > 0\}$$

and abbreviations $B^n(r) = B^n(0, r)$, $\mathbb{B}^n = B^n(1)$, $S^{n-1}(r) = S^{n-1}(0, r)$, and $S^{n-1} = S^{n-1}(1)$.

Recall that the volume of the n-dimensional unit ball can be expressed as

$$V_n = \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2} + 1)},$$

where Γ is Euler gamma function. Recall also that if by ω_{n-1}, we denote the $(n-1)$-dimensional measure of the sphere S^{n-1}, then

$$\omega_{n-1} = nV_n.$$

A Möbius transformation in complex plane is a mapping of the form

$$z \mapsto \frac{az + b}{cz + d},$$

where a, b, c, and d are complex constants such that $ad \neq bc$.

The homeomorphism $f : D \longrightarrow D'$ is called conformal, where D and D' are domains in $\mathbb{R}^n$, if f is $C^1(D)$, if $J_f(x) \neq 0$ for all $x \in D$, and if $|f'(x)h| = |f'(x)|\,|h|$ for all $x \in D$ and $h \in \mathbb{R}^n$. An anticonformal is the complex conjugate of a conformal mapping.

If D and D' are domains in $\overline{\mathbb{R}^n}$, a homeomorphism $f : D \longrightarrow D'$ is conformal if its restriction to $D \setminus \{\infty, f^{-1}(\infty)\}$ is conformal. This definition of conformal homeomorphism is preferable to others, since if f is conformal, then f^{-1} is conformal as well because $J_f(x) \neq 0$.

Let D and D' be domains in $\overline{\mathbb{R}}^n$. We call a C^1-homeomorphism $f : D \longrightarrow D'$ *sense-preserving* (orientation-preserving) if $J_f(x) > 0$ for all $x \in D \setminus$

$\{\infty, f^{-1}(\infty)\}$. If $J_f(x) < 0$ for all $x \in D \setminus \{\infty, f^{-1}(\infty)\}$, then we call f *sense-reversing* (orientation-reversing).

The mapping $f : D \longrightarrow D'$ is regular at $x \in D$ when f is differentiable at x and $J_f(x) \neq 0$.

A bijective mapping $f : G_1 \to G_2$ between domains G_1 and G_2 in $\mathbb{R}^n$ is a diffeomorphism if f and f^{-1} are continuously differentiable. Note that if f is a diffeomorphism, then $J_f(x) \neq 0$ for all $x \in G_1$.

Let D and D' be domains in $\mathbb{R}^n$. The Hessian of a function $f : D \longrightarrow \mathbb{R}^n$ is defined to be

$$\det\left(\left[\frac{\partial^2 f}{\partial x_i \partial x_j}\right]_{i,j=1}^{n}\right).$$

A function $f : D \longrightarrow \mathbb{R}$ is called harmonic if $f \in C^2(D)$ and

$$\Delta f := \sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i^2} = 0.$$

The operator Δ is called Laplacian. The set of harmonic functions forms a linear space.

A complex valued function $f : \Omega \longrightarrow \mathbb{C}$, where Ω is domain in $\mathbb{R}^n$, is called harmonic if both of its real and imaginary parts are harmonic functions in Ω.

Let X, Y be domains in $\mathbb{R}^n$. If a function $f : X \longrightarrow Y$ belongs to $C^2(X)$ and if each of its coordinate functions is harmonic, then we say that f is harmonic.

Note that the real and imaginary parts of the complex analytic function are harmonic functions.

Note that if f is harmonic and g analytic, then the composition $f \circ g$ is a harmonic function, but the composition $g \circ f$ need not be harmonic in general. Note also that all conformal mappings are analytic and that, in the plane, all analytic functions are harmonic. Additionally, in the plane, each harmonic function is locally the real (imaginary) part of some analytic function.

The Harnack inequality for positive harmonic function $u : G \longrightarrow (0, \infty)$ states that for every $s \in (0, 1)$, there is $C \geq 1$ such that

$$\max_{z \in B_x} u(z) \leq C \min_{z \in B_x} u(z),$$

holds, whenever $B^n(x, r) \subseteq G$ and $B_x = \overline{B}^n(x, sr)$.

Green's function on the unit ball is defined by

$$G(x, y) = \Phi(y - x) - \Phi(|x|(y - x^*)), \quad x, y \in \mathbb{B}^n, \quad x \neq y.$$

Here, ω_{n-1} is the $(n-1)$-dimensional measure of S^{n-1}, $x^* = |x|^{-2}x$, and the function

$$\Phi(x) = \begin{cases} -\frac{1}{2\pi}\log|x|, & n = 2, \\ \frac{1}{n(n-2)\omega_{n-1}}\frac{1}{|x|^{n-2}}, & n \geq 3, \end{cases}$$

defined for $x \in \mathbb{R}^n$, $x \neq 0$ is the fundamental solution of the Laplace's equation.

Recall that

$$P(x, \xi) = \frac{1 - |x|^2}{|x - \xi|^n}$$

is the Poisson kernel for the unit ball in $\mathbb{R}^n$.

Recall also that a continuous function $u : G \longrightarrow \mathbb{R}$ defined on a domain $G \subset \mathbb{C}$ is subharmonic if for all $z_0 \in G$, there exists $\varepsilon > 0$ such that

$$u(z_0) \leq \frac{1}{2\pi}\int_0^{2\pi} u(z_0 + re^{it})\,dt \text{ for } 0 < r < \varepsilon. \tag{*}$$

A continuous function $u : G \longrightarrow \mathbb{R}$ defined on a domain $G \subset \mathbb{C}$ is superharmonic if $-u$ is subharmonic.

Let a and b are reals such that $a < b$. For path $\alpha : [a, b] \longrightarrow \mathbb{R}^n$, we define its length $l(\alpha)$ as the supremum of values of the form

$$\sum_{i=1}^{n} |\alpha(t_i) - \alpha(t_{i-1})|$$

where $n \in \mathbb{N}$ and

$$a = t_0 < \cdots < t_n = b.$$

If $l(\alpha) < \infty$, then we say that α is a rectifiable path.

If α is a rectifiable path, then there is a unique path $\alpha^0 : [0, l(\alpha)] \longrightarrow \mathbb{R}^n$ such that there is a continuous increasing function $h : [a, b] \longrightarrow [0, l(\alpha)]$ such that $\alpha = \alpha^0 \circ h$ and for any $t \in [0, l(\alpha)]$, $l(\alpha^0|_{[0,t]}) = t$. We call the path α^0 a normal representation of α.

Let $I \subseteq \mathbb{R}$ is interval and $f : I \longrightarrow \mathbb{R}^n$. We say that f is absolutely continuous if for every $\varepsilon > 0$, there is $\delta > 0$ such that for every $m \in \mathbb{N}$ and for every sequence $(a_1, b_1), \ldots, (a_m, b_m)$ of pairwise disjoint subintervals of I, we have that

$$\sum_{i=1}^{n} |a_i - b_i| < \delta \Rightarrow \sum_{i=1}^{n} |f(a_i) - f(b_i)| < \varepsilon.$$

A Jordan curve or simple closed path in metric space (X, d) is a continuous (with respect to the metric d) mapping $\gamma : [0, 1] \longrightarrow X$ such that for all $x, y \in [0, 1]$ with $x < y$,

$$\gamma(x) = \gamma(y) \Leftrightarrow x = 0 \wedge y = 1.$$

A plane domain is Jordan if its boundary is a Jordan curve. For higher dimensions n, we say that a domain in $\mathbb{R}^n$ is a Jordan domain if its boundary is homeomorphic to the unit sphere.

We denote the α-dimensional Hausdorff measure of a set $F \subset \mathbb{R}^n$ by $\Lambda_\alpha(F)$. Recall that

$$\Lambda_\alpha^\delta(F) = \inf\{\sum_{i=1}^{\infty} d(U_i)^\alpha\},$$

where the infimum is taken over all countable coverings of F by sets U_i with $d(U_i) < \delta$, then set $\Lambda_\alpha(F) = \lim_{\delta \to 0} \Lambda_\alpha^\delta(F)$. The Hausdorff dimension of a set F is defined as follows

$$\dim_H(F) = \inf\{\alpha : \Lambda_\alpha(F) < \infty\}.$$

The β-dimensional Hausdorff content is defined to be

$$\Lambda^\beta(E) = \inf\left\{\sum_{i=1}^{\infty} r_i^\beta\right\},$$

where the infimum is taken over all coverings of $E \subseteq \mathbb{R}^n$ with countably many (Euclidean) balls of radii r_i .

Let $G \subset \mathbb{R}^n$ be a domain, and let $w : G \longrightarrow (0, \infty)$ be continuous. For given $x, y \in G$, let

$$d_w(x, y) = \inf\{l_w(\gamma) : \gamma \in \Gamma_{xy},\ l(\gamma) < \infty\}, \quad l_w(\gamma) = \int_\gamma w(\gamma(z))|dz|.$$

It turns out that this is a metric on G. If a length-minimizing curve exists, it is called a geodesic.

For a proper subdomain G of $\mathbb{R}^n$, the quasihyperbolic length of a rectifiable curve γ in G is given by

$$l_k(\gamma) = \int_\gamma \frac{|dz|}{d(z, \partial G)}.$$

The quasihyperbolic distance between points x and y from G is the infimum of quasihyperbolic lengths over all rectifiable curves in G joining x and y.

For an easy reference, we also record the Bernoulli inequality,

$$\log(1 + as) \leqslant a \log(1 + s), \quad a \geqslant 1,\ s > 0.$$