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Edmundo Capelas de Oliveira

Solved Exercises in Fractional Calculus

Springer

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Dedicated to Ivana,
my pretty woman.

Foreword I

The fractional calculus is an old topic born soon after the fundamental papers by Leibniz and Newton on integer-order calculus at the end of the seventeenth century. Indeed, we find contributions involving names of renowned researchers, among them, Euler, Fourier, Abel, Liouville, and Riemann. After the first international conference held in 1974, there was a significant increase in technical papers, and starting since the 1990s in textbooks. Nowadays, we can say that consolidated applications are found in diverse areas of science, but fractional calculus is still not considered a regular discipline in most universities. Faced with this scenario, the book of Prof. Capelas, written in a simple and objective way, aims to contribute to fill this gap. The book, after a brief historical introduction, exhibits an extensive list of solved and proposed exercises presented in a tutorial way in 4 chapters, along with a final chapter of applications. The reader can surely appreciate the way on which the solutions are presented: first the statements, then suggestions, and finally the detailed answers are found. May I suggest this book to both beginners and advanced scholars in fractional calculus.

Bologna, Italy

Francesco Mainardi
University of Bologna

Foreword II

Is it possible to extend the order of the derivatives and integrals to any number irrational, fractional, or complex? Indeed, Gottfried Leibniz conceived that concept in 1695! This initial brilliant idea remained in the area of pure mathematics for centuries. However, recently fractional calculus (FC) attracted the attention of many mathematicians, physicists, and engineers for the developments of practical implementations. In fact, during the recent years FC emerged as an important scientific area, stimulated by the findings and applications, not only in physics and engineering, but also in areas such as finance, economy, or biology. This progress motivated the development of a large volume of studies and publishing of many books addressing distinct aspects of FC, going from pure mathematics up to research in specific topics of applied sciences. In spite of these efforts toward the dissemination of FC to a wider audience, we verify that there is still the need for educational works that guide readers without experience in the field. The new book entitled "Solved exercises in fractional calculus" is the first work purely dedicated to this exercise, that is, to introduce researchers, or even the general public, to the surprising tools and shining results produced under the umbrella of FC. This area, simultaneously old and new, with three hundred years, but rediscovered only recently, will become certainly more accessible when following this new educational work that establishes a compromise both between basic theory and advanced research, and mathematics and applications. This book will certainly enlighten and delight readers.

Porto, Portugal

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Foreword III

Fractional calculus has received considerable interest in recent years. It is natural therefore that students and researchers may be interested in the subject and wish to enter this research area. However, to advance in the study of something new in mathematics, nothing is more useful than solving problems. This book, by Prof. Edmundo Capelas de Oliveira, a leading specialist in fractional calculus, is undoubtedly the book that every beginner in an area would like to find. The book covers the main themes of fractional calculus with a smart strategy: after an introduction to a given topic, problems are proposed, followed by suggestions for solving problems, succeeded by a section with the solution of the problems, and ended with proposed problems. No doubt this is a very valuable book in the subject of fractional calculus.

Campinas, Brazil

Jayme Vaz Jr
University of Campinas

Preface

Arbitrary order calculus or fractional calculus is of considerable interest, as it opened new and fascinating horizons to science, freeing it from the dependence of the sensible world, limited to the integer order calculus. Although we cannot physically imagine the true appearance of fractional derivatives, their study deserves attention because it sharpens our intuition and imagination.[1]
The future of fractional calculus as a discipline is assured.[2]

It is common in almost all higher level courses in exact sciences and technology that the student, when facing for the first time the discipline calculus, will find it difficult to follow, as most students arrive at the university without the knowledge and maturity necessary to study it. For this reason, many authors write books with solved and proposed exercises by means of which they try to overcome such lack of basis by providing the basic knowledge necessary for the student to be able to learn the new subject. However, one must keep in mind that this is not enough. The student must do his part, as such books are usually just a kind of guide to be used in parallel with an adequate textbook on the discipline.

It is not different here. This is a book dedicated to solved and proposed exercises, but with the one peculiarity that discipline fractional calculus has yet to be a fully consolidated part of standard curricula. Fractional calculus is as old as the traditional, integer-order calculus proposed independently by Newton and Leibniz at the end of the seventeenth century. However, for one reason or another, fractional calculus had its development accelerated only from the second half of the twentieth century, after the first international conference on the subject. Nowadays, fractional calculus is completely consolidated and is a subject of research in several fields of knowledge, particularly in problems in which the memory effect is present, as in the cases of anomalous diffusion, polymer studies, and image processing, just to mention a few examples.

[1] Adapted from: G. Arcidiacono, *Spazio, Iperspazi, Frattali*, Di Renzo Editore, Roma, (1993).

[2] Miklós Mikolás, *On the recent trends in the development theory and applications of fractional calculus and its applications*, Lectures Notes in Mathématices, **457**, 357–375, Springer-Verlag, (1974).

With these facts in mind, we have elaborated this work. It does not intend to replace books dedicated to the theory of fractional calculus or its specificities, nor is it a compendium on the subject. It does have the intention of being an introductory text that we deem necessary for accompanying a possible course on fractional calculus. Besides, it can also be used to introduce students to this fascinating subject, aiming at new studies and research, as its wide range of applications provide young people a new and motivating field of work.

We believe that it is important for teachers to be able to answer the classic question posed by (mainly first year) students when one starts teaching a new subject: "What's the use?" or "Where will I need this?" Thus, we have tried to make it clear, for each of the topics addressed, where it can be used or applied—in the developments presented in later chapters, in one or more proposed problems or, in some cases, by making specific reference to works in which that topic played an auxiliary or more fundamental role.

After these few lines justifying the work, we can make a brief list of the topics addressed. In passing, we mention that our choices were made with a view to a future textbook for a discipline of fractional calculus to which this book might be a companion.

The historical introduction occupies the entire first chapter. In the second chapter, we present the gamma function as a generalization of the factorial function, together with a few other important functions: the incomplete gamma, beta, error, complementary error, hypergeometric and confluent hypergeometric functions. We also introduce Meijer's G-function, the generalized Wright function and Fox's H-function. The third chapter is dedicated to the Mittag-Leffler function, known as the queen of functions of fractional calculus, as well as to its variations, with emphasis on the Wright and Mainardi functions. The fourth chapter deals with integral transforms, with emphasis on Laplace, Fourier, and Mellin transforms. Then, in the fifth chapter, we introduce the concepts of fractional integral and fractional derivative, highlighting the Riemann–Liouville, Caputo and Hadamard versions and mentioning some possible generalizations. In the sixth chapter, by means of real problems and applications, we discuss some fractional differential, integral, and integro-differential equations and present some new results not included in previous chapters.

As our main objective is the solution of exercises, chapters two–five bring a list of proposed exercises. At first, we present only the statements, as we believe that students should try to discern the best way to address a given problem. After that, we present suggestions for solving each exercise and then their complete solutions. With this method, we try to make it clear that there are three distinct phases to pass through in order to obtain any particular solution. Moreover, at the end of chapters two–five, we present a further list of exercises which, we believe, can be solved by anyone who could understand the preceding developments. An appendix involving the essentials of Mellin–Barnes integral closes the book.

I would say that this is a small part of the studies in the past 10 years, but very important. A thank you to my former students, all of them teachers in various places in Brazil and abroad. Also, I would like to thank my colleagues

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Campinas, Brazil Edmundo Capelas de Oliveira

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