Cauchy's *Calcul Infinitésimal*

An Annotated English Translation

Dennis M. Cates

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For my wonderful wife, Dawne.

Translator's Preface

Augustin-Louis Cauchy (1789–1857) is responsible for a great many publications, but among his most important works is his textbook of 1823, *Résumé des leçons données a l'École royale polytechnique sur le calcul infinitésimal* (Summary of Lectures given at the Royal Polytechnic School on the Infinitesimal Calculus). He wrote the text, usually shortened to *Calcul infinitésimal*, while teaching at the *École polytechnique* in Paris, one of the most prestigious institutes of higher education in all of Europe,[1] to support the analysis course he taught at the school. Cauchy's *Calcul infinitésimal,* translated here in its entirety, is his first full-length book devoted to calculus. In a collection of lecture notes designed for his students at the École Polytechnique, Cauchy presents his differential and integral calculus in a highly efficient and well-organized manner. In this text, he builds a complete and solid foundation for calculus from which the current subject is based.

The *Calcul infinitésimal* text is composed of forty relatively brief lectures which begin with his historic definition of the limit, then quickly progress in rigorous and methodical fashion through all of the important topics and theorems necessary to construct what is essentially the modern version of this subject. His forty lectures present material in an order nearly identical to today's calculus textbooks, beginning with differential calculus, stepping through integral calculus, and ending with a study of infinite series and Taylor's Theorem. *Calcul infinitésimal* is important because it presents a cohesive, consistent, and comprehensive package for the systematic use and rigorous supporting theory of the calculus—a package based firmly and squarely upon reason. Cauchy methodically steps through result after result, introducing relevant precise definitions as needed. Basic theorems are rigorously proven one after the other, while more advanced theorems are derived in turn—always basing the argument upon earlier results. His definitions for most all of the important concepts in the subject are the same modern definitions we use today, with only relatively minor modifications in some cases which occur later in the 19th and early 20th centuries.

[1] The École Polytechnique is still in operation today as a member of France's elite *Grande écoles*. It was established in 1794 by the French Revolutionary government to train scientists and engineers and is still regarded as one of the most elite and selective institutions in Europe.

Many of Cauchy's ideas presented in *Calcul infinitésimal* were not new and had been suggested in an ad hoc way by many mathematicians over the years long before Cauchy was even born. The basic concept of the limit had been discussed among leading mathematicians as far back as the 17th century, but without success. Jean-Baptiste le Rond d'Alembert (1717–1783) had pushed for the idea to be integrated into a theory of the calculus as recently as the late 18th century, but once again his idea did not catch on. As has occurred so many times through the history of science and mathematics, the early 19th century needed just the right brilliant individual to coalesce all the ideas then floating in the air. An individual who also possessed the ability to combine these ideas with the strength of mathematics and proper reason to form a consistent and rigorous theory. It was Augustin-Louis Cauchy who would be this person. Cauchy was the individual who had the insight and creativity to finally bring in the precision of formulation of the limit concept with the correct mathematical muscle allowing him to place the calculus on a formal footing. It would be Cauchy who would manage to form a universal rigorous foundation for the calculus. Indeed, Cauchy's work set a new standard of mathematical rigor in Europe for the remainder of the 19th century and would finally, after nearly 150 years of attempts, place the calculus on firmly defendable ground.

The setting in which Cauchy found himself in this world is one of the most interesting in modern history. He was born at the dawn of the French Revolution into a family tightly associated with the monarchy of France. Although his family lived through a great deal of financial and political turmoil during his childhood, his father's position in the government eventually allowed the young Cauchy access to highly influential people. Two of these individuals were the prominent and important French statesmen and scientists, Pierre-Simon de Laplace (1749–1827) and Joseph-Louis Lagrange (1736–1813). Both became family friends and recognized Cauchy's mathematical talent early. They helped place him in the best schools in Paris, ultimately to include the École Polytechnique, where Cauchy would then have André-Marie Ampère (1775–1836) as a tutor for his analysis and other courses. Knowing today the importance of the mathematical and scientific contributions from even these three mentors of Cauchy, the life into which this young man was launched is simply an incredible one.

A Few Notes Regarding Cauchy's Original Works:

Although real, single-variable functions are the primary focus of Cauchy's *Calcul infinitésimal* lectures, he spends time exploring how to extend his development to multiple variable and complex functions. Among many key results in the text, Cauchy will lay out all the common tools needed to utilize the calculus. Employing his precise limit definition, Cauchy will define both the derivative and the definite integral as limits themselves—firmly setting them both on a solid and rigorous foundation. His definition of the definite integral will ultimately serve as a model for Bernhard Riemann's (1826–1866) more modern version decades later. Along with this definition, Cauchy will demonstrate the existence of the definite integral independent of the derivative, a pioneering step forward in the field. Within *Calcul infinitésimal*, Cauchy will provide rigorous, limit-based proofs for the General Mean

Value Theorems of the Derivative and the Definite Integral, as well as the Fundamental Theorem of Calculus and many others. Convergence properties of infinite series are covered in depth near the end of the text, including standard convergence tests and theorems. It is here where Cauchy will again seriously break from many of his contemporaries and require the convergence of an infinite series to be demonstrated before any further claims regarding its properties can be made. Cauchy will prove Taylor's Theorem and show the Lagrangean attempt to rigorize the calculus is inadequate.[2]

As stated earlier, Cauchy published an enormous amount of mathematics, second only to Leonhard Euler (1707–1783). However, of main interest to us here are two of the first full-length books Cauchy wrote in the early 1820s. The first of these two is, of course, our *Calcul infinitésimal*, published in 1823. The second is a book published two years earlier, *Cours d'analyse de l'École royale polytechnique* (Analysis Course at the Royal Polytechnic School). Within this earlier text, Cauchy lays out much of the analysis fundamentals required to develop his calculus fully, including his definitions of the limit and of continuity. But, the text stays away from the strictly calculus topics of the derivative and the integral. He initially planned for this earlier work to include two volumes, the first volume of which was published in 1821 with the title *Analyse algébrique* (Algebraic Analysis). For various reasons, mostly related to disagreements with the *Conseil d'instruction* (Board of Instruction)[3] over the expected content of the analysis course to be taught at the École Polytechnique, the second volume of this work was never produced. It is now customary to refer to the published first volume simply as *Cours d'analyse* instead of *Analyse algébrique*; but, the two titles are used interchangeably, even by Cauchy himself. Due to the wealth of analysis results found in this text, Cauchy will reference it often in *Calcul infinitésimal* and clearly expected his students to be familiar with the text, or to refer to it heavily for themselves during the analysis course he was teaching.

In the year 1882, well after Cauchy's death, a monumental effort was begun to compile all of Cauchy's works into a single collection. The title of this compilation is *Œuvres complètes d'Augustin Cauchy* (Complete Works of Augustin Cauchy). It consists of 27 volumes, each published on different dates. Incredibly, this undertaking was not completed until the 1970s. Cauchy's *Cours d'analyse* is located in Series

[2] Many in the mathematics community of the early 19th century thought most any function could be expanded into a Taylor series, and that the derivative of the function was simply the coefficient of the second (or linear) term of this expansion. Lagrange himself was a leading proponent of this theory and taught this algebra-based theory of the calculus at the École Polytechnique during its early years. His important 1797 (and later editions) textbook, *Théorie des fonctions analytiques* (The Theory of Analytic Functions), was derived from these lectures. Indeed, Lagrange taught the same analysis course that Cauchy would later inherit when young Augustin-Louis began his tenure at the school in 1815 at the age of 26.

[3] This is the administrative body at the École Polytechnique overseeing the content and duration of each course in its curriculum. There was a near-constant struggle between Cauchy desiring a rigorous class which was full of foundational analysis, and the Conseil d'Instruction who desired a practical course shorter in length and one with more applications. This struggle would cause a serious headache for Cauchy for much of his time at the École Polytechnique.

II, Tome III of this huge collection, published in 1897. His *Calcul infinitésimal* is found in Series II, Tome IV, published in 1899.

A Few Technical Notes Regarding This Translation:

This translation generally follows the overall page layout and formatting of the 1899 version, as the original 1823 version is quite cramped and unnecessarily difficult to read, especially when it comes to Cauchy's formulas and equations. The content, emphasis, and special use of typeset of the two versions are nearly identical, but when any variations occur Cauchy's original 1823 words and choice of text style are almost always used here, the version under Cauchy's control. The only rare exceptions occur when the 1899 version more clearly presents Cauchy's message and intent. Any footnotes included by Cauchy himself (of which there are only two) are fully replicated in this translation.

A short errata sheet was published with the original 1823 text. This sheet has been included here; however, to improve the readability of the translation, each of the noted errors in the text has been corrected. Wherever one of these corrections occurs, a footnote has been included to identify the change. There are multiple additional errors throughout the original 1823 text beyond those listed in its errata sheet. These, too, have been corrected and noted as they arise. The 1899 edition contains its share of duplicate and new errors of its own. To be complete in this translation, these have also been documented.

The 1899 reprint added several new footnotes throughout the text in places where Cauchy references a result from his 1821 *Cours d'analyse*. These new footnotes mark where in the *Œuvres complètes d'Augustin Cauchy* this reference can be found. Cauchy makes a total of seventeen such direct references to his earlier work throughout the *Calcul infinitésimal* text, twelve of which have added footnotes in the 1899 edition. Since the original 1821 version of *Cours d'analyse* is widely available,[4] and once one understands where the 1897 reprint of this book is located within the *Œuvres complètes d'Augustin Cauchy*, these new footnotes are not usually very helpful. As a result, all have been excluded from this translation to reduce unnecessary clutter.

The editors of the *Œuvres complètes d'Augustin Cauchy* chose to append one of Cauchy's research papers, "On the Formulas of Taylor and Maclaurin," to the very end of the 1899 reprint of his *Calcul infinitésimal*. Although this new work is not a part of Cauchy's original *Calcul infinitésimal*, it too is translated and included as Appendix D. Additional appendices to this book include translations of selected excerpts from Cauchy's *Cours d'analyse* which hold particular interest.

In an effort to maintain the feel of an early 19th-century text, nearly all of the original mathematical notation and symbolism have been retained in the translation, with only a small number of exceptions. Included in these exceptions is Cauchy's use of a low-positioned dot to represent multiplication, as an example $u.\frac{1}{v}$, or to represent the differential of some functions, as an example $d.\sin x$. Cauchy himself is not consistent with the usage of this low-positioned dot in either situation, and its

[4] BnF Digital Library, *Bibliotèque nationale de France—Gallica*, www.bnf.fr.

usage is abandoned in the 1899 reprint. There are a few other minor modifications, including an occasional use or removal of parentheses, but the number of changes from the original 1823 text is quite small.

Notes have been included throughout the translation to help the reader in locations where Cauchy is particularly terse or unclear with his mathematics. As a modern reader progresses through these pages, one needs to keep in mind they are a collection of his lecture notes, developed and published to benefit his students. It must be assumed there was plenty Cauchy would have been explaining in the classroom as he went through this material in person from his blackboard that does not show up here in his text. These added footnotes help to fill that gap. Important mathematical results Cauchy has achieved, or misconceptions he may have had at the time are also noted to aid the reader get a sense of history in the making and a feel for how difficult this subject was to develop, even for one of the best mathematicians in the early 19th century. Occasional historical comments are included to remind the reader of the rich history surrounding the development of calculus and of the eventful time in Europe when Cauchy wrote the text in Paris.

Without losing Cauchy's message or his characteristic style, punctuation has been adjusted on occasion, but no attempt has been made to significantly modify Cauchy's early 19th-century wording. The primary goal of this translation has been to present Cauchy's original text in as true a form as possible, so the reader has a sense of what it may have been like to actually sit in an analysis class of Mr. Cauchy's in the 1820s.

A Few Final Thoughts:

Cauchy's lectures provide an excellent vehicle for the lifelong mathematics student to gain an appreciation for the historical development of analysis and the calculus. It may also serve as a particularly valuable supplement to a traditional calculus or real analysis text for those who desire a way to create more texture in a conventional calculus class by introducing original historical sources. As the modern reader will see, Cauchy leaves much for his students and future readers to discover and demonstrate for themselves within the text.

It is hoped you will find the experience of full immersion into Cauchy's world fruitful and personally rewarding. Take your time and leisurely work through Cauchy's lectures line by line to gain a complete appreciation for the development of his calculus. Witness firsthand how Cauchy struggles in spots and makes errors in others. In this manner, you will experience the thrill and excitement one gains by walking the same path as the master, Mr. Cauchy. Enjoy the journey.

—D. Cates, Ph.D.

SUMMARY OF LECTURES

GIVEN

AT THE ROYAL POLYTECHNIC SCHOOL

ON

THE INFINITESIMAL CALCULUS,

By Mr. Augustin-Louis CAUCHY,

Engineer of Bridges and Roads,
Professor of Analysis at the Royal Polytechnic School,
Member of the Academy of Sciences,
Knight of the Legion of Honor.

FIRST VOLUME.

PARIS,
THE ROYAL PRINTING OFFICE.

1823.

Original title page of 1823 text.

FOREWORD.

This work,[5] undertaken on the request of the Board of Instruction of the Royal Polytechnic School,[6] offers a summary of the lectures that I gave to this school on the infinitesimal calculus. It will be composed of two volumes[7] corresponding to the two years which form the duration of the course. I publish the first volume today divided into forty lectures, the first twenty of which comprise the differential calculus, and the last twenty a part of the integral calculus. The methods that I follow differ in several respects from those which are found expressed in the works of similar type. My main goal has been to reconcile the rigor, which I have made a law in my *Analysis Course*, with the simplicity which results from the direct consideration of infinitely small quantities. For this reason, I thought obliged to reject the expansion of functions by infinite series, whenever the series obtained are not convergent;[8] and I saw myself forced to return the formula of Taylor to the integral calculus, and this formula can only be admitted as general so long as the series that it contains is found reduced to a finite number of terms and supplemented by a definite integral. I am aware that the illustrious author of the *Analytical Mechanics* has taken the formula in question[9] for the basis of his theory of *derived functions*.[10] But, despite all

[5] This is the original *AVERTISSEMENT* included in the 1823 edition.

[6] Known as the *Conseil d'instruction.*

[7] Cauchy had been at constant odds with the Conseil d'Instruction over the content and duration of the analysis course for years. They much preferred a shorter course with more applications. In the summer of 1824, following the publication of *Calcul infinitésimal* (the first volume of this work), the printing of the lectures for the second volume was begun (the first thirteen lectures were printed), but then suddenly halted. The reason surrounding this interruption remains unclear, but likely was due to the Conseil d'Instruction's displeasure with the material Cauchy had thus far developed. In any event, the printing was never resumed and the complete second volume was never published.

[8] Cauchy breaks from many mathematicians up to his time by requiring the convergence of an infinite series be demonstrated before any claim of its properties could be made.

[9] The author Cauchy is referring to is one of his mentors, Joseph-Louis Lagrange (1736–1813). The formula is that of Taylor.

[10] The phrase *derived function* had previously been used and established by Lagrange, but today *derivative function* is more commonly employed.

the respect that such a grand authority commands, the majority of mathematicians[11] are now in accordance to recognize the uncertainty of the results which we can be led to by the employment of divergent series, and we add that in several cases, the Theorem of Taylor seems to provide the expansion of a function by convergent series, even though the sum of the series differs fundamentally from the proposed function (*see* the end of the thirty-eighth lecture). Moreover, those who will read my work, will be convinced, I hope, that the principles of the differential calculus and its most important applications, can be easily explained without the intervention of series.

In the integral calculus, it seemed necessary to demonstrate generally the existence of *integrals* or *primitive functions* before making known their various properties. To achieve this, it was first necessary to establish the notion of *integrals taken between given limits*, or *definite integrals*.[12] These latter objects can sometimes be infinite or indeterminate, it being essential to study in which cases they maintain a unique and finite value. The simplest means of resolving the question are the employment of *singular definite integrals* which are the subject of the twenty-fifth lecture. Moreover, among the infinite number of values that we can attribute to an indeterminate integral, there exists one which merits our particular attention and that we have named *principal value*. The consideration of singular definite integrals and those of the principal values of indeterminate integrals are very useful in the solution of a large group of problems. We deduce a great number of general formulas that work for the determination of definite integrals and are similar to those that I gave in a report presented to the Institute in 1814. We will find in lectures thirty-four and thirty-nine a formula of this type applied for the evaluation of several definite integrals, some of which were already known.

[11] Cauchy uses the term *géomètres* here, a term that would have referred to any mathematician during his time.

[12] The integral was considered not nearly as important as the derivative by most mathematicians of Cauchy's time, generally viewing it simply as the inverse of the derivative. Cauchy's definite integral existence proof is important because it demonstrates the independence of the integral from the derivative.

ERRATA.

PAGES.	LINES.	ERRORS.	CORRECTIONS.
28.	27.	$\boldsymbol{L}(x)$	$\boldsymbol{l}(x)$
Ibid.	28.	$\boldsymbol{L}(x+h) - \boldsymbol{L}(x)$	$\boldsymbol{l}(x+h) - \boldsymbol{l}(x)$
37.	28.	$\boldsymbol{L}\left(\frac{x}{\mathbf{y}}\right)$	$\boldsymbol{L}\left(\frac{x}{y}\right)$
47.	23.	**,**	,
Ibid.	29.	d^2x	$d^2 x$
56.	26.	$d_x^n u,$	$v d_x^n u,$
68.	2 and 12.	$F(r, 0, 0, 0..)$	$F(r)$
Ibid.	*Ibid.*	$F(r, s, 0, 0..)$	$F(r, s)$
Ibid.	*Ibid.*	$F(r, s, t, 0..)$	$F(r, s, t)$

This is the original errata sheet included in the 1823 publication. All of these errors have been corrected in this translation along with added footnotes wherever a change has been made. The corrections on pages 37 and 47 (these are page references to Cauchy's original text) are due to slightly different characters or symbols being used. The variations between what was actually used for the printing and the corrected versions are slight—perhaps an odd typeset in use at the time.

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CONTAINED IN THIS VOLUME.

INTEGRAL CALCULUS.

END OF THE TABLE.

APPENDICES.

[13]This addendum is not listed in the original 1823 TABLE OF CONTENTS but is included in the published text.