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Approximation Theory and Algorithms for Data Analysis

Springer

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Preface

This textbook offers an elementary introduction to the theory and numerics of approximation methods, combining classical topics of approximation with selected recent advances in mathematical signal processing, and adopting a constructive approach, in which the development of numerical algorithms for data analysis plays an important role.

Although the title may suggest otherwise, this textbook is not a result of the current hype on *big data science*. Nevertheless, both classical and contemporary topics of approximation include the analysis and representation of functions (e.g. signals), where suitable mathematical tools (e.g. Fourier transforms) are essential for the analysis and synthesis of the data. As such, the subject of *data analysis* is a central topic within approximation, as we will discuss in further detail.

Prerequisites. This textbook is suitable for undergraduate students who have a sound background in linear algebra and analysis. Further relevant basics on numerical methods are provided in Chapter 2, so that this textbook can be used by students attending a course on numerical mathematics. For others, the material in Chapter 2 offers a welcome review of basic numerical methods. The text of this work is suitable for courses, seminars, and distance learning programs on approximation.

Contents and Standard Topics. The central theme of approximation is the characterization and construction of best approximations in normed linear spaces. Readers are introduced to this standard topic (in Chapter 3), before approximation in Euclidean spaces (in Chapter 4) and Chebyshev approximation (in Chapter 5) are addressed. These are followed by asymptotic results concerning the approximation of univariate continuous functions by algebraic and trigonometric polynomials (in Chapter 6), where the asymptotic behaviour of Fourier partial sums is of primary importance. The core topics of Chapters 3-6 should be an essential part of any introductory course on approximation theory.

More Advanced Topics. Chapters 7-9 discuss more advanced topics and address recent developments in modern approximation and its relevant applications. To this end, Chapter 7 explains the basic concepts of signal approximation using Fourier and wavelet methods. This is followed by a more comprehensive introduction to multivariate approximation by meshfree positive definite kernels in Chapter 8. The material in Sections 8.4-8.5 provides more recent results concerning relevant aspects of convergence, stability, and update strategies for kernel-based approximation. Moreover, Section 8.6 presents basic facts on kernel-based learning. Lastly, Chapter 9 focuses on mathematical methods of computerized tomography, exploring this important application field from the viewpoint of approximation. In particular, new convergence results concerning the approximation of bivariate functions from Radon data are proven in Section 9.4.

For those who have studied Chapters 3-6, any of the three more advanced topics in Chapters 7-9 could seamlessly be included in an introductory course on approximation. Nevertheless, it is strongly recommended that readers first study the Fourier basics presented in Sections 7.1-7.4, since much of the subsequent material in Chapters 8-9 relies on Fourier techniques.

Exercises and Problem Solving. Active participation in exercise classes is generally an essential requirement for the successful completion of a mathematics course, and a (decent) course on approximation is certainly no exception. As such, each of the Chapters 3-9 includes a separate section with exercises. To enhance learning, readers are strongly encouraged to work on these exercises, which have different levels of complexity and difficulty. Some of the exercise problems are suitable for group work in class, while others should be assigned for homework. Although a number of the exercise problems may appear difficult, they can be solved using the techniques explained in this book. Further hints and comments are available on the website

`www.math.uni-hamburg.de/home/iske/approx.en.html`.

Biographical Data. To allow readers to appreciate the historical context of the presented topics and their developments, we decided to provide footnotes, where we refer to those whose names are linked with the corresponding results, definitions, and terms. For a better overview, we have also added a name index. The listed biographical data mainly relies on the online archive *MacTutor History of Mathematics* [55] and on the free encyclopedia *Wikipedia* [73], where more detailed information can be found.

Acknowledgement. The material of this book has grown over many years out the courses on approximation and mathematical signal processing that I taught at the universities of Hamburg (Germany), Lund (Sweden), and Padua (Italy). I thank the participating students for their constructive feedback, which has added great didactical value to this textbook. Moreover, I would like to thank my (post)doctoral students Dr Adeleke Bankole, Dr Matthias Beckmann, Dr Benedikt Diederichs, and Niklas Wagner for their careful proofreading. Additional comments and suggestions from Dr Matthias Beckmann and Dr Benedikt Diederichs concerning conceptional and didactical aspects as well as the technical details of the presentation are gratefully appreciated. Last but not least, I would like to thank Dr Martin Peters (SpringerSpektrum, Heidelberg) for his support and encouragement, which led to the initiation of the book project.

Hamburg, October 2018

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